Multi-product Monopolist and Information Design*

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Abstract

We study the information design and pricing decision of a multi-product monopolist who faces a buyer with unit demand. Under omniscient disclosure, we show that one profit-maximizing menu reveals the identity of the highest quality product and allocates that product, provided a type-specific quality threshold is met. This menu remains optimal under private disclosure if the buyer's value has constant differences in quality and type, or if the value has non-decreasing differences in quality and type and the threshold type is independent of quality. We also characterize the conditions under which an ex-ante and non-discriminatory posted price mechanism is optimal.

Keywords: Information design; Optimal mechanism; Multi-product monopolist; Omniscient disclosure; Private disclosure; Price discrimination

JEL codes: D82, D83, L15

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1 Introduction

In many practical applications, the seller and the buyers know the quality distributions of the products but are equally unsure of the exact level of quality. Under these circumstances, the seller may strategically conduct experiments to reveal the quality of the products. This consideration is especially relevant if the seller is a monopolist with multiple differentiated products, and each buyer demands at most one product. The joint decision of information design and pricing determines the seller's ability to screen the buyers and reap profits.

As a concrete example, consider the decision of Roche, a Swiss multinational holding healthcare corporation. The pharmaceutical division of the company sells two antiviral medicines, Tamiflu and Xofluza, for treatment of seasonal influenza.[1] Since both oral treatments have been approved by the Food and Drug Administration (FDA), the healthcare company and the potential buyers are approximately equally informed of the effectiveness of the two types of drugs against influenza A and B. Nevertheless, they do not know the exact level of quality, which determines their respective valuation for each type of drug. Each potential buyer demands at most one type of drug[2] and Roche may conduct or fund Phase IV clinical trials to further reveal the quality of each type of drug. Our goal is to analyze the profit-maximizing menu of experiments and mechanisms for a seller in a similar situation as the Swiss healthcare company.

As an illustration, consider a monopolist who offers two products, 1 and 2, to a single buyer who demands at most one product.[3] Both the seller and the buyer's values can depend on the product quality. For a product of quality q, the seller's value is $c(q)$ and the buyer's value is $v(q, \theta)$, where θ is the buyer's private type and is independent of q_1 or q_2. The buyer knows their type, but the seller only knows

[1] See https://www.cdc.gov/flu/treatment/whatyoushould.htm for more details on the two types of drugs as well as the FDA approval.

[2] Existing studies show that combining Xofluza with Tamiflu does not result in superior clinical outcomes for hospitalized patients with influenza (e.g., Kumar et al., 2022).

[3] An alternative interpretation is that there are copies of the two products to be sold to a unit measure of buyers. The quality realization of different copies of products are independent, and so are the type realization across buyers.

$\theta \sim U[0,1]$. It is common knowledge that product 1 is of deterministic quality, with $q_1 = 1$, and product 2 is of uncertain quality, with $q_2 \sim U[0,2]$. Both the seller and the buyer are expected utility maximizers with quasilinear preferences. In order to maximize their expected profit, the seller can devise a menu of verifiable experiments and selling mechanisms through which to sell the two products. The buyer first chooses a preferred combination of experiment and mechanism and then makes their purchasing decision after observing the outcome of the chosen experiment, which is either publicly or privately observed.

First consider the case of $v(q,\theta) = q\theta$ and $c(q) = \frac{1}{2}q$. If the seller chooses no disclosure, then the buyer/seller's expected value for the two products are identical. To earn the maximum profit, the seller excludes any buyer type $\theta \le \frac{3}{4}$, sells either product to any type $\theta > \frac{3}{4}$ and leaves zero rent to the cutoff type $\theta = \frac{3}{4}$. This can be achieved by posting prices $p_1 = p_2 = \frac{3}{4}$, which yields an expected profit of $\text{EP}^{\text{no}} = \frac{1}{16}$. Suppose instead the seller conducts a verifiable experiment to reveal whether $q_2 \ge q_1 = 1$ and posts prices $p_1 = \frac{3}{4}$ and $p_2 = \frac{9}{8}$. Given this menu of experiment and mechanism, the buyer never purchases any product if $\theta \le \frac{3}{4}$, purchases product 1 if $\theta > \frac{3}{4}$ and $q_2 < 1$, and purchases product 2 if $\theta > \frac{3}{4}$ and $q_2 \ge 1$. The seller's expected profit from this menu is $\text{EP}^* = \frac{5}{64} > \frac{1}{16} = \text{EP}^{\text{no}}$. It can be shown that this menu maximizes the seller's expected profit, irrespective as to whether the experimental outcome is publicly observed or privately observed by the buyer.

With the above value specification, we see it is optimal for the seller to post a non-discriminatory price for each product before conducting any experiment. The following specification suggests this is not always the case. Consider the setting of $v(q,\theta) = q + \theta$ and $c(q) = \frac{1}{2}q$. An analogous argument shows the seller can improve upon the no disclosure benchmark by conducting a verifiable experiment to reveal whether $q_2 \ge q_1 = 1$ and posting prices $p_1 = \frac{5}{4}$ and $p_2 = \frac{13}{8}$, which yields an expected profit of $\text{EP}^{\text{disclose}} = \frac{85}{128}$. Suppose instead the seller offers a menu of experiments and corresponding posted prices $(\{E(\eta)\}_{\eta\in[0,\frac{1}{4}]}, \{p_1(\eta), p_2(\eta)\}_{\eta\in[0,\frac{1}{4}]})$, where $E(\eta)$ reveals whether $q_2 \ge 2 - 4\eta$ and $p_1(\eta) = \frac{5}{4}$, $p_2(\eta) = 2 - \frac{3}{2}\eta$. Given this menu of experiments and mechanisms, the buyer's optimal choice of experiment and purchasing decision depends on their type. If $\theta > \frac{1}{4}$, the buyer chooses experiment $E(\frac{1}{4})$ and purchases product 1 if $q_2 < 1$ and purchases product 2 if $q_2 \geq 1$. If $\theta < \frac{1}{4}$, the buyer chooses experiment $E(\theta)$ and only purchases product 2 if $q_2 \geq 2 - 4\theta$. The seller's expected profit from this menu is $\mathrm{EP}^* = \frac{2}{3} > \frac{85}{128} = \mathrm{EP}^{\mathrm{disclose}}$. It can be shown that this menu maximizes the seller's expected profit, irrespective as to whether the experimental outcome is publicly observed or privately observed by the buyer.

There are three observations from this numerical example with two different value specifications. First, in both cases, the seller can improve their expected profit through strategic information disclosure. Intuitively, by conducting an experiment to reveal which product is of higher quality realization, the seller can induce a more efficient allocation and thus create more surplus to extract from. Second, it is sometimes but not always optimal for the seller to post a non-discriminatory price for each product before conducting any experiment. Intuitively, for $v(q, \theta) = q\theta$ and $c(q) = \frac{1}{2}q$, the seller wants to sell the higher quality product to the buyer if $\theta > \frac{3}{4}$, which is independent of the quality realization. For $v(q, \theta) = q + \theta$ and $c(q) = \frac{1}{2}q$, the seller wants to sell the higher quality to the buyer if $\theta > \frac{1}{2} - \frac{1}{4}q$, which depends on the quality realization. Consequently, price discrimination is of no additional value in the first case, but can be combined with information design to facilitate surplus extraction from the buyer in the second case. Third, in both cases, a profit-maximizing menu reveals the identity of the higher quality product and allocates that product to the buyer, provided a type-specific quality threshold is met. Moreover, this menu is optimal regardless of signal observability to the seller.

The rest of the paper serves to extend and strengthen the three observations in a general setting. We consider a monopolist seller with multiple products of distinct quality for sale to a single buyer with ex-ante unit demand. Both the seller and the buyer know the joint quality distribution but neither knows the exact realization. For a product of quality q, the seller's reservation value is $c(q)$ and the buyer's value is $v(q, \theta)$, where θ is the buyer's type. We suppose that both the seller and the buyer share the common prior of θ, which is independent of the joint quality distribution, but that only the buyer observes its realization. The seller's goal is to devise a menu of experiments and mechanisms that maximizes their expected profit. We make a few assumptions about the buyer's value function $v(q, \theta)$ and the virtual value function

$\varphi(q,\theta) := v(q,\theta) - c(q) - \frac{1-F(\theta)}{f(\theta)} v_\theta(q,\theta)$. Specifically, we suppose $v(.,.)$ is strictly increasing in both q and θ. In addition, we suppose $\varphi(.,.)$ is strictly increasing in θ, and is strictly increasing in q above the threshold quality.[4] Roughly speaking, these assumptions guarantee that the products are vertically differentiated. Regarding signal observability, we allow for both omniscient and private disclosures and analyze the two cases separately.[5]

For the case of omniscient disclosure, we characterize one profit-maximizing menu of experiments and mechanisms. Specifically, we show that the seller can achieve the optimal profit, subject to the buyer's incentive constraints, by offering a type-specific experiment that reveals the identity of the highest quality product and allocating that product to the buyer, provided the highest quality product meets the type-specific threshold quality. We further show that the set of experiments in this menu is the Blackwell least informative experiment for every buyer type among the collection of experiments that can achieve the optimal profit. The importance of the latter characterization is two-fold: first, if the seller could not design arbitrarily informative experiments, then what we characterize is the set of minimally informative experiments that still enables the seller to achieve the optimal profit; second, we show if the seller could achieve the same profit under private disclosure as under omniscient disclosure, then they could do so by conducting the set of Blackwell least informative experiments. In other words, the set of experiments that we characterize is the one most robust to signal informativeness and observability that enables the seller to achieve the optimal profit.

For the case of private disclosure, we impose an additional assumption that the quality realizations of different products are mutually independent. Under this additional assumption, we show that the seller can achieve the same profit as in the case of omniscient disclosure if either (1) the buyer's value has constant differences in product quality and buyer type *or* (2) the buyer's value has non-decreasing differences in product quality and buyer type and the threshold type is independent of

[4] The threshold quality represents the minimum quality at which the seller is willing to relinquish a product to the buyer (see Equation 1 for details).

[5] In an omniscient disclosure, both the seller and the buyer observe the outcome of the experiments (Bergemann and Morris, 2016). In a private disclosure, only the buyer observes the experimental outcome (Li and Shi, 2017; Kolotilin et al., 2017).

product quality.[6]

Then we explore the conditions under which price discrimination is not beneficial to the seller. We show it is profit-maximizing to post a non-discriminatory price for each product ex ante if either (a) the threshold type is independent of product quality *or* (b) the buyer's value is identical for every quality-threshold type pair. With a single product, the two conditions are also necessary. With multiple products, there are other value/virtual functions such that an ex-ante and non-discriminatory posted price mechanism is optimal for *some* quality distributions, but the two cases are the only possibilities where an ex-ante and non-discriminatory posted price mechanism is optimal for *all*/*all independent* quality distributions under omniscient/private disclosure. Finally, we analyze the conditions under which information design is not beneficial in the single-product scenario. For both the cases of omniscient and private disclosures, we show the seller cannot improve upon the no disclosure benchmark if the threshold type is independent of product quality. For the case of omniscient disclosure, we additionally show this condition is necessary, that is, the seller can strictly improve upon the no disclosure benchmark when this condition fails.

Taken together, our analysis sheds light on the role of information design and price discrimination for a multi-product monopolist.

Related Literature Our paper is at the intersection of two strands of literature: optimal information disclosure (Rayo and Segal, 2010; Kamenica and Gentzkow, 2011; Kolotilin et al., 2017) and optimal pricing and mechanism design (Mussa and Rosen, 1978; Myerson, 1981).

There is a growing literature that combines optimal pricing with information design, but almost all existing and concurrent works focus on a single product. One set of papers study the optimal nondiscriminatory disclosure policy and corresponding pricing scheme at which to sell one product to one or more buyers, such as Lewis and Sappington (1994), Ottaviani and Prat (2001), Johnson and Myatt (2006) and Bergemann and Pesendorfer (2007). The key insight of these papers is that whether

[6] The threshold type represents the minimum buyer type to which the seller is willing to relinquish a product of given quality (see Equation 2 for details).

the buyer has private information, if the seller is constrained to use a single non-discriminatory disclosure policy, then either full or no disclosure is optimal.[7] Some other works study a similar problem, such as Eső and Szentes (2007), Li and Shi (2017), Chen (2019) and Guo, Li and Shi (2022), but allow for private disclosure. Their key message is that if the seller is constrained to use orthogonal disclosure policies, that is, the information the seller discloses is independent of the buyer's private information, then full disclosure is optimal with continuous buyer types, while partial disclosure may be optimal otherwise. Our paper also studies a similar problem under discriminatory disclosure but differs from these two sets of works in two key aspects. First and foremost, we suppose the seller offers *multiple* products, each of distinct quality. The key difference from the single-product scenario is that even if the seller posts a price for each product, a unit-demand buyer's action space is no longer binary, that is, the buyer can choose between purchasing any one of the products or not purchasing a product, instead of just whether to purchase or not to purchase. As Kolotilin et al. (2017) point out, the number of actions significantly affects the complexity of the buyer's incentive constraints, especially under private disclosure. In addition, we suppose the seller can conduct experiments only with respect to the product quality, which is assumed to be independent of the buyer's private type. In other words, in contrast to most previous works that focus on differences in the buyer's taste, our model particularly fits the application of a monopolist seller with multiple vertically differentiated products.[8]

Two closely related papers are the concurrent works by Smolin (2023) and Wei and Green (2023). Similar to our setting, they also study the optimal discriminatory disclosure policy with respect to the product's attributes and corresponding pricing menu of a monopolist seller. Specifically, Smolin (2023) considers a single product with multiple attributes and analyzes the structure of the optimal menu of disclosure policy and pricing rule under private disclosure, whereas Wei and Green (2023)

[7]One implicit assumption is that the seller commits to a *deterministic* disclosure rule. Krähmer (2020) shows this result may not hold if the seller is allowed to randomize over information structures.

[8]In a concurrent work, Bergemann, Heumann and Morris (2023) also consider the situation where a monopolist can offer various products of distinct quality to screen the buyer, but they assume the seller has full control over the quality of these products and conducts experiments with respect to the buyer's private type.

focus on a single product with a single attribute and put restrictions on the buyer's valuation, so that the optimal pricing menu exhibits reverse price discrimination. By comparison, we analyze the case of multiple products, each with a single attribute. As explained in the previous paragraph, the multi-product setting is sharply different from the single-product scenario because the incentive constraints are significantly more complicated, especially under private disclosure. Under this framework, we characterize the seller's profit-maximizing menu of experiments and selling mechanisms under both omniscient and private disclosures. Moreover, our analysis shows that whether the threshold type depends on the product quality would affect the seller's ability to screen the buyer and helps explain the seemingly diverse results of Smolin (2023) and Wei and Green (2023) in the single-product, single-attribute scenario (Proposition 4).

In terms of economic insight, our analysis is also related to the literature on optimal bundling (Maskin and Riley, 1984; Yang, 2023). In our setting, for a given quality profile, the optimal mechanism is always *non-screening* (Haghpanah and Siegel, 2022). The key question for the seller is how to bundle the different quality realizations into experiments so that the buyer's incentive constraints are respected under omniscient or private disclosure.

The rest of the paper is organized as follows: Section 2 describes the model setup, formulates the seller's profit-maximization problem and introduces the assumptions. Section 3 presents one optimal menu of experiments and mechanisms under omniscient disclosure, and shows this menu remains optimal under private disclosure if the buyer's value has constant differences in product quality and buyer type. Section 4 analyzes the conditions under which an ex-ante and non-discriminatory posted mechanism is optimal and Section 5 concludes. All of the proofs are relegated to the appendix.

2 Model

This section presents the model setup. Section 2.1 introduces the model preliminaries. Section 2.2 first specifies the incentive constraints on the buyer and then formulates the seller's problem. Section 2.3 discusses the assumptions.

2.1 Model Setup

A monopolist seller has m products of different quality for sale to a single buyer. Given product quality q, the seller's reservation value for any product is $c(q)$, whereas the buyer's valuation for any product is $v(q,\theta)$, where θ is the buyer's private type. Let $\boldsymbol{q}=(q_1,q_2,\ldots,q_m)$ be the quality profile of the m products, where q_i is the quality of product i. Suppose $q_i \in [\underline{q}_i, \bar{q}_i]$ is a random variable, commonly unknown to both the seller and the buyer. Further suppose the seller and the buyer share the same common prior about $\boldsymbol{q}$, with joint distribution function G and marginal distribution function G_i. Let $Q := \times_{i=1}^m [\underline{q}_i, \bar{q}_i]$ be the set of possible quality profiles. We assume G is absolutely continuous and has full support on Q, but allow G_i to be degenerate, that is, it is possible that $\underline{q}_i = \bar{q}_i$ for some product i.

Let the buyer's type $\theta \in [\underline{\theta}, \bar{\theta}]$, which is assumed to be their private information. Suppose the seller and the buyer share the same common prior about θ, which can be represented by an absolutely continuous density function $f(.) > 0$ on $[\underline{\theta}, \bar{\theta}]$, with corresponding distribution function F. Further suppose the buyer has ex-ante unit demand (which is formalized in Assumption 3) and both the seller and the buyer are expected utility maximizers with quasilinear preferences. To be precise, consider any allocation-payment pair $(\boldsymbol{x}, t)$, where $x_i \geq 0$ is the probability the buyer receives product i and t is the expected transfer/payment from the buyer to the seller. For $\sum_{i=1}^m x_i \leq 1$, the buyer's expected utility for any realized quality profile $\hat{\boldsymbol{q}}$ is given by

$$U_B(\boldsymbol{x}, t; \theta) = \sum_{i=1}^m x_i v(\hat{q}_i, \theta) - t$$

The seller's expected utility for this realized quality profile is given by

$$U_0(\boldsymbol{x}, t) = t - \sum_{i=1}^m x_i c(\hat{q}_i)$$

Experiment(s) The seller can design a family of experiments $\{E_\gamma\}_{\gamma\in\Gamma}$ to screen the buyer and disclose information of the realized quality profile. One statistical experiment/information structure E consists of a measurable signal realization space S and a family of probability mass/density functions $\{\pi(.|\boldsymbol{q})\}_{\boldsymbol{q}\in Q}$ over S. Any one experiment can be arbitrarily informative of the realized quality profile. For instance, in the case of no disclosure, $S = \{s_0\}$ is a singleton set and $\pi(s = s_0|\boldsymbol{q}) = 1$, for any $\boldsymbol{q} \in Q$. In the case of full disclosure, $S = Q$ and $\pi(s = \boldsymbol{q}|\boldsymbol{q}) = 1$, $\pi(s = \boldsymbol{q}'|\boldsymbol{q}) = 0$, for any $\boldsymbol{q} \in Q$ and $\boldsymbol{q}' \neq \boldsymbol{q}$.

Selling Mechanism Let $X = \{\boldsymbol{x}\colon\ 0 \leq x_i \leq 1,\ 1 \leq i \leq m\}$ be the set of possible allocations. Given an experiment $E = (S, \pi)$, we consider a general class of tariff mechanisms

$$T : X \times S \to \mathbb{R},$$

which, given signal realization s, specifies payment $T(\boldsymbol{x}, s)$ the buyer has to make for each random bundle $\boldsymbol{x} \in X$. If the seller does not offer a bundle $\boldsymbol{x_0} \in X$ given signal realization s, then $T(\boldsymbol{x_0}, s) = +\infty$.

Timing The game proceeds as follows:

1. The seller designs a menu consisting of a set of experiments and tariff mechanisms which can be conditioned on the signal realizations *or* the buyer's reported signal realizations, that is, $\{E_\gamma, T(E_\gamma)\}_{\gamma\in\Gamma}$, and commits to this menu.

2. The quality profile $\boldsymbol{q}$ is realized according to G, unobserved to both the seller and the buyer. The buyer's type θ is realized according to F, which is privately observed.

3. The buyer chooses one combination of experiment and tariff mechanism $(E, T(E))$ from the menu.

4. Given the buyer's choice of experiment $E = (S, \pi)$, signal $s \in S$ is realized and is publicly observed by both the seller and the buyer *or* privately observed by the buyer.

5. Given signal realization s, the buyer makes their purchasing decision by choosing allocation $\boldsymbol{x} \in X$ with payment $T(\boldsymbol{x}, s)$ or the outside option of no purchase.

Our goal is to find a menu of experiments and tariff mechanisms that maximizes the seller's expected profit.

Remark 1: There are two clarifications regarding the game rule.

(1) With multiple products, the optimal profit the seller can attain when the signal realization is publicly observed (i.e., omniscient disclosure) can be quite different from the case where the signal realization is privately observed (i.e., private disclosure), and we will analyze the two cases separately.

(2) The buyer can leave the market with no purchase anytime during the game.

2.2 Buyer's Incentive Constraints and Seller's Problem

We start with the setting of omniscient disclosure. Since the signal realization is publicly observable, the revelation principle still holds in the presence of information design. Specifically, for any menu of experiments and tariff mechanisms $\{E_\gamma, T(E_\gamma)\}_{\gamma\in\Gamma}$, there exists a menu of direct experiments $\{E(\theta) = (S(\theta), \pi(\theta))\}_{\theta\in[\underline{\theta},\bar{\theta}]}$ and mechanisms $\{\boldsymbol{x}(\theta, s), t(\theta, s)\}_{\theta\in[\underline{\theta},\bar{\theta}], s\in S(\theta)}$ (where $x_i(\theta, s)$ is the probability that a type θ buyer receives product i given signal realization s and $t(\theta, s)$ is the corresponding total payment of this buyer) that implements the same outcome in a truthful revelation.

Let $U(\theta', s; \theta)$ be the expected utility of the buyer who is a true type θ, whereas they report *and act* as being type θ' and the public signal realization is s. Specifically, for $\sum_{i=1}^m x_i(\theta, s) \leq 1$, $\forall \theta, \theta' \in [\underline{\theta}, \bar{\theta}]$ and $s \in S(\theta')$, we have

$$U(\theta', s; \theta) = \sum_{i=1}^m x_i(\theta', s)\mathbb{E}[v(q_i, \theta)|s, E(\theta')] - t(\theta', s)$$

For notational simplicity, we write $U(\theta, s) := U(\theta, s; \theta)$, which represents the expected utility of a type θ buyer under truthful reporting and the public signal realization is

s. Then the buyer's incentive constraints are defined as follows:

$$\int_{s\in S(\theta)} U(\theta,s)\pi(\theta,s)ds \geq \int_{s\in S(\theta')} \max\{U(\theta',s;\theta),0\}\pi(\theta',s)ds, \quad \forall \theta,\theta' \in [\underline{\theta},\bar{\theta}] \quad (\text{IC}_{\text{om}})$$

$$U(\theta,s) \geq 0, \quad \forall \theta \in [\underline{\theta},\bar{\theta}] \text{ and } s \in S(\theta) \quad (\text{IR})$$

Since the buyer can quit with no purchase anytime during the game, the set of participation constraints (IR) is imposed ex post and requires that for *every* signal realization of the chosen experiment, the buyer is willing to take their consumption-payment bundle instead of the outside option. For the same reason, when evaluating a candidate misreport, the buyer takes into account of the possibility that they may quit with no purchase under some signal realization, which explains the term $\max\{U(\theta',s;\theta),0\}$ in the set of incentive constraints (IC_{om}).

From Section 2.1, the seller's expected utility/profit from menu of direct experiments and mechanisms $(\{E(\theta)\},\{x(\theta,s),t(\theta,s)\})$ is given by

$$\text{EP} = \mathbb{E}_\theta\left[\int_{s\in S(\theta)}\left(t(\theta,s) - \sum_{i=1}^{m} x_i(\theta,s)\mathbb{E}[c(q_i)|s,E(\theta)]\right)\pi(\theta,s)ds\right] \quad (\text{EP})$$

Consequently, the seller's problem under omniscient disclosure is given by

- **Profit maximization under omniscient disclosure**: maximizes (EP), subject to (IC_{om}) and (IR).

Next we consider the case of private disclosure. Since the signal realization is privately observed by the buyer, sequential screening may yield a strictly better outcome for the seller and the revelation principle generally does not hold (Courty and Li, 2000). For simplicity and direct comparison, however, we still restrict attention to a menu of direct experiments and mechanisms in this setup. In particular, if the seller could achieve the same outcome as in the setting of omniscient disclosure, then we know this restriction is without loss.[9]

[9] A similar approach is adopted by Wei and Green (2023), where they introduce an auxiliary benchmark such that the quality is perfectly revealed to both parties after the seller chooses a mechanism.

For the case of private disclosure, let $U(\theta', s'; \theta, s)$ be the buyer's expected utility when they report and act as being type θ' and the signal realization is $s' \in S(\theta')$, whereas their true type is θ and the true signal realization is $s \in S(\theta')$. Specifically, for $\sum_{i=1}^m x_i(\theta, s) \leq 1$, $\forall \theta, \theta' \in [\underline{\theta}, \bar{\theta}]$ and $s, s' \in S(\theta')$, we have

$$U(\theta', s'; \theta, s) = \sum_{i=1}^{m} x_i(\theta', s') \mathbb{E}[v(q_i, \theta) | s, E(\theta')] - t(\theta', s')$$

For notational simplicity, we write $U(\theta, s) := U(\theta, s; \theta, s)$. Then the seller is subject to the following set of incentive constraints of the buyer: for any $\theta, \theta' \in [\underline{\theta}, \bar{\theta}]$,

$$\int_{s \in S(\theta)} U(\theta, s) \pi(\theta, s) ds \geq \int_{s \in S(\theta')} \sup_{s' \in S(\theta')} \{U(\theta', s'; \theta, s), 0\} \pi(\theta', s) ds \tag{IC$_{\text{pr}}$}$$

In other words, (IC$_{\text{pr}}$) strengthens (IC$_{\text{om}}$) by ensuring that the buyer always has incentives to choose the experiment designed for their type, even when they could jointly misreport their type and the observed signal realization. In particular, for $\theta' = \theta$, (IC$_{\text{pr}}$) implies that the buyer always has incentives to truthfully report the signal realization, conditional on their truthful report of the type. That is,

$$U(\theta, s) \geq U(\theta, s'; \theta, s), \quad \forall \theta \in [\underline{\theta}, \bar{\theta}] \text{ and } s, s' \in S(\theta)$$

The seller's problem under private disclosure is given by

- **Profit maximization under private disclosure**: maximizes (EP), subject to (IC$_{\text{pr}}$) and (IR).

2.3 Assumptions

This subsection introduces five assumptions: the first four will be maintained throughout the analysis, while the last assumption will be utilized for some results on private disclosure.

The first two assumptions concern the buyer's value and virtual value for different quality products.

Assumption 1 (Vertical Differentiation)**.** *The value function satisfies*

(1) $v(q, \theta)$ *is continuous and strictly increasing in* (q, θ).

(2) $v_\theta(q, \theta)$ *exists and is continuous in* (q, θ).

Roughly speaking, Assumption 1 states that the different products are vertically differentiated. In other words, whatever their type, the buyer places a higher value on a higher quality product. Moreover, the buyer's value is increasing in their type.

Let $\varphi(q, \theta) := v(q, \theta) - c(q) - \frac{1-F(\theta)}{f(\theta)} v_\theta(q, \theta)$ be the buyer's *virtual value*. We define the *threshold quality* of a type θ buyer as

$$q^\star(\theta) = \begin{cases} \max_i \bar{q}_i & \text{if } \varphi(q, \theta) \leq 0 \text{ for all } q \in [\min_i \underline{q}_i, \max_i \bar{q}_i] \\ \inf\{q \in [\min_i \underline{q}_i, \max_i \bar{q}_i] : \varphi(q, \theta) > 0\} & \text{otherwise} \end{cases} \tag{1}$$

A related definition is the *threshold type* of a quality q product, which is given by

$$\theta^\star(q) = \begin{cases} \bar{\theta} & \text{if } \varphi(q, \theta) \leq 0 \text{ for all } \theta \in [\underline{\theta}, \bar{\theta}] \\ \inf\{\theta \in [\underline{\theta}, \bar{\theta}] : \varphi(q, \theta) > 0\} & \text{otherwise} \end{cases} \tag{2}$$

Intuitively, the threshold quality represents the minimum quality at which the seller is willing to relinquish a product to a type θ buyer, whereas the threshold type represents the minimum buyer type to which the seller is willing to relinquish a product of quality q. In our context, there is a mapping between the threshold quality and the threshold type, and we will work with the more convenient definition based on the question of interest.

The next assumption imposes analogous conditions to Assumption 1 on the buyer's virtual value.

Assumption 2 (Vertical Differentiation for Virtual Value)**.** *The virtual value satisfies*

(1) $c(q)$ *is continuous in* q.

(2) $\varphi(q,\theta)$ is strictly increasing in θ, and is strictly increasing in q for $q \geq q^\star(\theta)$.

Item (2) of Assumption 2 only imposes monotonicity for product quality above the threshold quality because, in the optimal menu of experiments and mechanisms, no product with quality less than $q^\star(\theta)$ would be allocated to a type θ buyer.

The third assumption formalizes the idea that the buyer has ex-ante unit demand.

Assumption 3 (Unit Demand)**.** *For any of the seller's choice of experiments and mechanisms, $\sum_{i=1}^m x_i(\theta, s) \leq 1$, for any θ, s.*

This assumption is necessary but not sufficient for ex-post unit demand, which states that the buyer will receive either nothing or one unit of one of the products. In terms of economics, the seller may offer random allocations. For instance, if $m = 2$, then an allocation such as $(0.5, 0.5)$ is allowed. On the other hand, we will show in Section 3 that the optimal menu of experiments and mechanisms features ex-post unit demand under every signal realization.

The next assumption guarantees that whatever their type, the buyer would share the same common prior G of the quality profile.

Assumption 4 (Independence of Product Quality Profile and Buyer Type)**.** *The quality profile $\boldsymbol{q} = (q_1, q_2, \dots, q_m)$ and the buyer's private type θ are independent.*

The final assumption requires quality independence of different products and will only be used in analyzing the profit-maximization problem under private disclosure.

Assumption 5 (Independent Product Quality)**.** *The quality of the m products $q_1, q_2, \dots, q_m$ are mutually independent.*

One implication of this assumption is $\mathbb{E}[h(q_i)|q_i = \max_l q_l] \geq \mathbb{E}[h(q_i)|q_j = \max_l q_l]$ for any non-decreasing function $h(.)$. For instance, we can take $h(.) = v(., \theta)$. On the other hand, this inequality may not hold without mutual independence. [10]

[10] A counter-example is as follows: suppose $m = 2$ and $q_1 \in [2, 3]$, $q_2 \in [1, 4]$. Further suppose (q_1, q_2) has a close to 0.5 probability around $(2, 1)$ and a close to 0.5 probability around $(3, 4)$. Then for the identity function $h(q) = q$, we have $\mathbb{E}[q_1|q_1 \geq q_2] < \mathbb{E}[q_1|q_2 \geq q_1]$.

3 Optimal Menu of Experiments and Mechanisms

This section analyzes one profit-maximizing menu of direct experiments and mechanisms. In Section 3.1, we focus on omniscient disclosure and characterizes one optimal menu of direct experiments and mechanisms. We further show that the set of experiments is the Blackwell least informative one for every buyer type among any set of experiments that can achieve profit maximization. In Section 3.2, we consider private disclosure and show that when the buyer's value is additively separable/has constant differences in product quality and buyer type, then the menu of experiments and mechanisms identified in Section 3.1 remains optimal under this alternative disclosure setup.

3.1 Omniscient Disclosure

Throughout this subsection, we suppose the signal realization is publicly observed by both the seller and the buyer and analyze the profit-maximization problem under omniscient disclosure.

Before characterizing the optimal menu of experiments and mechanisms, we first describe the optimal experiment for each buyer type. For a type θ buyer, we introduce an imaginary product 0 with quality $q_0(\theta) := q^\star(\theta)$ and let $S^*(\theta) = \{s_0^*(\theta), s_1^*(\theta), \ldots, s_m^*(\theta)\}$ be a signal space with $m+1$ signal realizations. Now let $Q(\theta) := \{q_0(\theta)\} \times Q$ be the collection of possible quality profiles for a type θ buyer.[11] Further let $Q_i^*(\theta) = \{\boldsymbol{q}(\theta) \in Q(\theta) : q_i(\theta) \geq \max_{0 \leq j < i} q_j(\theta), q_i(\theta) > \max_{i < j \leq m} q_j(\theta)\}$ be the collection of quality profiles in $Q(\theta)$ where product i is of the highest realized quality with the largest index. For any $\boldsymbol{q} \in Q_i^*(\theta)$, let $\pi^*(\theta, s_i^*(\theta)|\boldsymbol{q}) = 1$. Consider the following type-specific experiment:

$$E^*(\theta) = (S^*(\theta), \pi^*(\theta))$$

In other words, for a type θ buyer, the experiment reveals the identity of the highest quality product, provided the quality of that product is at least $q^\star(\theta)$. Otherwise,

[11] Recall from Section 2 that $Q := \times_{i=1}^m [\underline{q}_i, \bar{q}_i]$ is the collection of possible quality profiles for the m actual products.

the experiment only reveals the fact that each product is of a quality lower than $q^\star(\theta)$.

The following theorem states that under omniscient disclosure, the seller can achieve the optimal profit, subject to the buyer's incentive constraints, with the set of discriminatory experiments $\{E^*(\theta)\}$ and a set of discriminatory cutoff mechanisms.

Theorem 1 (Optimal Menu under Omniscient Disclosure)**.** *Under Assumptions 1 through 4, the seller can achieve the optimal expected profit, subject to (IC$_{om}$) and (IR), with the set of experiments $\{E^*(\theta)\}$ and the following set of direct mechanisms: for any $\theta \in [\underline{\theta}, \bar{\theta}]$ and $i = 1, 2, \dots, m$,*

$$x_i^*(\theta, s) = \begin{cases} 1 & \text{if } s = s_i^*(\theta) \\ 0 & \text{otherwise} \end{cases}$$

and

$$t^*(\theta, s) = \begin{cases} \mathbb{E}[v(q_i, \theta) | s_i^*(\theta)] - \frac{V_i(\theta)}{\pi^*(\theta, s_i^*(\theta))} & \text{if } s = s_i^*(\theta) \\ 0 & \text{if } s = s_0^*(\theta) \end{cases},$$

where $V_i(\theta) := \int_{\underline{\theta}}^{\theta} \mathbb{E}[v_\eta(q_i, \eta) | s_i^*(\eta)] \pi^*(\eta, s_i^*(\eta)) d\eta$.

Theorem 1 characterizes one solution to the profit-maximization problem under omniscient disclosure: for each buyer type, an experiment is offered to reveal the identity of the highest quality product, provided the quality of that product meets the buyer's type-specific threshold quality. When the latter condition is met, the realized highest quality product is allocated to the buyer.

To intuitively understand the optimal combination of disclosure policy and selling mechanism, we draw connection to the analysis on optimal bundling (Maskin and Riley, 1984; Yang, 2023). For simplicity, suppose $m = 2$ and consider a deterministic quality profile (q_1, q_2). Since we allow for random allocations, the key question is to identify the seller's optimal bundling policy, taking into account of the buyer's incentive constraints. An important observation is that under Assumption 2, the optimal selling mechanism *always* features no bundling *and* selling the higher quality product to the buyer, provided their type meets the threshold type of $\max_i q_i$. Consequently, if all realizations of $q_2 \geq \max\{q^\star(\theta), q_1\}$ are grouped into one signal, then the seller can still devise a proper payment scheme and induce the profit-maximizing allocation in an incentive compatible manner.

To ensure incentive compatibility, the following lemma shows that the threshold quality is weakly decreasing in buyer type, which leads to an increasing quality profile where each signal concerning the m actual products is sent.

Lemma 1 (Decreasing Threshold Quality)**.** *Under Assumption 2,*

$$q^\star(\theta) \geq q^\star(\theta'), \quad \textit{for any } \theta < \theta',$$

where $q^\star(\theta)$ is as defined in Equation (1).

Remark 2: There are three clarifications regarding the characterization in Theorem 1.

(1) This result does not rely on the product quality being mutually independent. Indeed, the connection with optimal bundling suggests that the seller's profit is maximized for *every* quality profile realization.

(2) Given the set of discriminatory experiments $\{E^*(\theta)\}$, the optimal allocation is unique, but the transfer rule is not unique for some valuation structures and quality distributions. In contrast to the single-product scenario, the envelope formula only pins down each buyer type's expected information rent across signal realizations. Nevertheless, the prescribed transfer rule ensures the set of incentive constraints ($\mathrm{IC}_{\mathrm{om}}$) and participation constraints (IR) are both respected across *all* valuation structures and quality distributions.

(3) The optimal disclosure policy is not unique. For instance, under omniscient disclosure, the seller can also achieve the optimal expected profit, subject to ($\mathrm{IC}_{\mathrm{om}}$) and (IR), through full disclosure and conditional posted prices

$$p_i^*(\boldsymbol{q}) = \begin{cases} v(q_i, \theta^\star(q_i)) & \text{if } q_i = \max_j q_j \\ +\infty & \text{otherwise} \end{cases},$$

where $\theta^\star(q)$ is the threshold type of quality q as defined in Equation (2).

With respect to item (3) of Remark 2, the following proposition illustrates the special role of $\{E^*(\theta)\}$.

Proposition 1 ($E^*(\theta)$ in terms of Blackwell Order)**.** *Suppose Assumptions 1 through 4 hold and that the seller can achieve the optimal profit, subject to (IC_{om}) and (IR), with a set of experiments $\{\tilde{E}(\theta)\}$ and a selling mechanism, then $\tilde{E}(\theta)$ is Blackwell more informative than $E^*(\theta)$, for almost all θ.*

In other words, $E^*(\theta)$ is the Blackwell least informative experiment for *every* buyer type among the collection of experiments that can achieve the optimal profit, subject to the buyer's incentive constraints. The importance of this result can be seen from two perspectives: first, we assume the seller could design arbitrarily informative experiments. If this is not the case, then $E^*(\theta)$ gives the minimally informative experiment for every buyer type that can still achieve the optimal profit; second, we will show in the next subsection that $\{E^*(\theta)\}$ is the set of experiments most robust to signal observability (i.e., omniscient or private disclosure) that can achieve the optimal profit.

3.2 Private Disclosure

In this subsection, we suppose the signal realization is privately observed by the buyer. Given our formulation in Section 2.2, the seller's profit under this alternative situation would be weakly lower than under omniscient disclosure. Our goal is to analyze whether and when the seller could achieve the same profit.

Lemma 2 (Sufficiency of $E^*(\theta)$ under Private Disclosure)**.** *Suppose Assumptions 1 through 4 hold and let $\Big(\{\tilde{E}(\theta) = (\tilde{S}(\theta), \tilde{\pi}(\theta))\}, \{\tilde{\boldsymbol{x}}(\theta, \tilde{s}(\theta)), \tilde{t}(\theta, \tilde{s}(\theta))\}\Big)$ be a menu of experiments and mechanisms that maximizes the seller's expected profit, subject to (IC_{om}) and (IR). If this menu also respects (IC_{pr}), then there exists a transfer rule $\{t(\theta, s_i^*(\theta))\}$ such that menu $(\{E^*(\theta)\}, \{\boldsymbol{x}^*(\theta, s_i^*(\theta)), t(\theta, s_i^*(\theta))\})$ also maximizes the seller's expected profit, subject to (IC_{pr}) and (IR).*

In other words, it suffices to focus on the menu of Blackwell least informative experiments $\{E^*(\theta)\}$ in analyzing whether and when the seller could achieve the same profit under private disclosure as under omniscient disclosure. A proof sketch is as follows: from item (2) of Remark 2, the optimal allocation under omniscient disclosure is unique, so $\tilde{\boldsymbol{x}}(\theta, .)$ must coincide with $\boldsymbol{x}^*(\theta, .)$ for every profile realization. Since the optimal allocation always involves no bundling and that $\tilde{E}(\theta)$ is Blackwell more informative than $E^*(\theta)$ for every buyer type, the seller could design the weighted average payment scheme for every signal realization $s_i^*(\theta)$ to achieve the same profit while respecting the buyer's incentive constraints.

Despite this simplification, it is still next to impossible to give a necessary and sufficient condition under which the seller could achieve the same profit with private disclosure as with omniscient disclosure. Such a task is challenging even with a single product. For instance, the "reverse price discrimination" condition identified by Wei and Green (2023) is sufficient but not necessary. With multiple products, there are at least two more complications: first, as remarked by Kolotilin (2017), with more than two actions, there are many ways to disobey a recommendation. For instance, in our context, the buyer could jointly misreport their type and the signal realization to purchase a product different from what the seller prescribes. Consequently, even if the "reverse price discrimination" condition holds for every product, the buyer may still have incentives to deviate. Indeed, whether a particular deviation is profitable for a buyer type crucially depends on the valuation structure as well as the joint quality distribution (see Example 2); second, as pointed out in item (2) of Remark 2, the optimal transfer rule under omniscient disclosure is not always unique (see Example 4), so even if the seller could not achieve the same profit with one particular transfer rule under private disclosure, it is unclear whether another transfer rule could fulfill the task.

Given these challenges, there are two directions to proceed: we could put restrictions either on the buyer's value/virtual value, or the common prior of joint quality distribution. We follow the first direction and identify two mutually exclusive and practically relevant valuation structures such that the seller can always achieve the same profit with private disclosure as with omniscient disclosure.

(1) The buyer's value is additively separable, or equivalently, has constant differences in product quality and buyer type.

(2) The buyer's value has non-decreasing differences in product quality and buyer type, and the threshold type is independent of product quality.

We focus on case (1) in the current subsection and defer case (2) to the next section, where we show the seller can achieve the goal by posting a non-discriminatory price for each product ex ante.

Before presenting the results for case (1), we formalize the idea of "additively separable valuation" and impose a corresponding stronger condition on the buyer's virtual value.

Assumption 1S (Additively Separable Valuation)**.** *$v(q, \theta) = a(q) + b(\theta)$, where $a(q)$ is absolutely continuous and strictly increasing in q, and $b(\theta)$ is continuously differentiable, strictly increasing and weakly concave in θ.*

Remark 3: A standard result from partial differential equations shows $v_{q\theta}(., .) = 0$ if and only if $v(., .)$ is of the form $a(q) + b(\theta)$. In other words, $v(., .)$ is additively separable if and only if it has constant differences in product quality and buyer type.

When $v(q, \theta) = a(q) + b(\theta)$, the virtual value $\varphi(q, \theta) = a(q) - c(q) + b(\theta) - \frac{1-F(\theta)}{f(\theta)} b'(\theta)$.

Assumption 2S (Strong Vertical Differentiation for Virtual Value)**.** *$\varphi(., .)$ satisfies*

(1) The inverse hazard rate $\frac{1-F(\theta)}{f(\theta)}$ is non-increasing in θ.

(2) $c(q)$ is absolutely continuous and non-decreasing in q, and $a(q) - c(q)$ is strictly increasing in q.

In the context of additively separable valuations, Assumption 2S is slightly stronger than Assumption 2. For instance, Assumption 2 only requires $b(\theta) - \frac{1-F(\theta)}{f(\theta)} b'(\theta)$ to be strictly increasing in θ, while Assumption 2S additionally requires the second term $\frac{1-F(\theta)}{f(\theta)} b'(\theta)$ to be non-increasing in θ. For the same reason, Assumption 1S requires $b(.)$ to be weakly concave so that the distribution function of $b(\theta)$ has the monotone hazard rate property whenever the distribution function of θ does so.

The following theorem shows that with an additively separable valuation, menu $(\{E^*(\theta)\}, \{\boldsymbol{x}^*(\theta, s_i^*(\theta)), t^*(\theta, s_i^*(\theta))\})$ remains optimal under private disclosure. In particular, the seller can achieve the same profit as with omniscient disclosure.

Theorem 2 (Optimal Menu with Additively Separable Valuation under Private Disclosure)**.** *Under Assumptions 1S, 2S, 3, 4 and 5, the seller can achieve the optimal expected profit, subject to (IC_{pr}) and (IR), with the set of discriminatory experiments $\{E^*(\theta)\}$ and the following set of direct mechanisms: for any $\theta \in [\underline{\theta}, \bar{\theta}]$ and $i = 1, 2, \ldots, m$,*

$$x_i^*(\theta, s) = \begin{cases} 1 & \textit{if } s = s_i^*(\theta) \\ 0 & \textit{otherwise} \end{cases}$$

and

$$t^*(\theta, s) = \begin{cases} \mathbb{E}[v(q_i, \theta)|s_i^*(\theta)] - \frac{V_i(\theta)}{\pi^*(\theta, s_i^*(\theta))} & \textit{if } s = s_i^*(\theta) \\ 0 & \textit{if } s = s_0^*(\theta) \end{cases},$$

where $V_i(\theta) := \int_{\underline{\theta}}^{\theta} \mathbb{E}[v_\eta(q_i, \eta)|s_i^*(\eta)]\pi^*(\eta, s_i^*(\eta))d\eta$.

Remark 4: There are three clarifications regarding the characterization in Theorem 2.

(1) In contrast to Theorem 1, this result crucially relies on the quality of the m products being mutually independent. As discussed after Assumption 5, this assumption guarantees that $\mathbb{E}[h(q_i)|q_i = \max_l q_l] \geq \mathbb{E}[h(q_i)|q_j = \max_l q_l]$ for any non-decreasing function $h(.)$.

(2) Our formulation of additively separable valuation encompasses the special case of $v(q, \theta) = q + \theta$ and $c(q) = c$, so this result generalizes the baseline result of Wei and Green (2023) to multiple independent products.

(3) Unless the transfer/payment is independent of the signal realization, full disclosure is never optimal under private disclosure, since the buyer has incentives to under-report the highest quality. Theorem 2 and Lemma 2 together illustrate that $\{E^*(\theta)\}$ is the set experiments most robust to signal observability that can achieve the seller's optimal profit.

The following lemma explains the special feature of the additively separable valuation structure that enables the seller to achieve the same profit with private disclosure as with omniscient disclosure.

Lemma 3 (Simplification of Incentive Constraints with $E^*(\theta)$ under Private Disclosure)**.** *Suppose Assumptions 1 through 4 hold and consider any direct mechanism* $\{\boldsymbol{x}(\theta, s_i^*(\theta)), t(\theta, s_i^*(\theta))\}$ *such that menu* $(\{E^*(\theta)\}, \{\boldsymbol{x}(\theta, s_i^*(\theta)), t(\theta, s_i^*(\theta))\})$ *maximizes the seller's expected profit, subject to (IC$_{om}$) and (IR). Further suppose* $U(\eta, s) \geq U(\eta, s'; \eta, s)$*, for some* $\eta \in [\underline{\theta}, \bar{\theta}]$ *and any* $s, s' \in S(\eta)$*.*

(1) If $v(q, \theta)$ *has non-decreasing differences in* (q, θ)*, then* $U(\eta, s; \theta, s) \geq U(\eta, s'; \theta, s)$*, for any* $\theta > \eta$ *and* $s, s' \neq s_0^*(\eta)$*.*

(2) If $v(q, \theta)$ *has non-increasing differences in* (q, θ)*, then* $U(\eta, s; \theta, s) \geq U(\eta, s'; \theta, s)$*, for any* $\theta < \eta$ *and* $s, s' \neq s_0^*(\eta)$*.*

Recall from Lemma 2 that it suffices to focus on the set of Blackwell least informative experiments $\{E^*(\theta)\}$ in analyzing whether the seller could achieve the same profit with private disclosure as with omniscient disclosure. Lemma 3 further simplifies the analysis of the buyer's incentive constraints under private disclosure when the set of experiments is $\{E^*(\theta)\}$. To intuitively understand this lemma, we restrict attention to additively separable valuations, where both conditions hold. As argued above, even if the set of (IC$_{\text{om}}$) constraints is satisfied, the buyer may profit from jointly misreporting their type and the signal realization under private disclosure, which means there are many ways for the buyer to disobey the recommendation/misreport the signal when there are multiple products. For instance, for a type θ buyer, if they misreport as being type θ' and the true signal realization is $s_1^*(\theta')$, then they may profit from reporting signal $s_2^*(\theta')$ and purchasing product 2, instead of truthfully reporting the signal and purchasing product 1. Roughly speaking, Lemma 3 states such a deviation would not be profitable if the buyer's value is additively separable. More precisely, if the buyer would truthfully report the signal realization conditional on their truthful report of the type, then they would still do so even if they misreport their type, except possibly for the case when the signal realization indicates no purchase.

Even with Lemma 3, the proof of Theorem 2 is challenging because with multiple products and no restriction on each product's quality distribution, the set of potentially profitable deviations is still large under private disclosure. We need to guarantee (1) conditional on their truthful report of the type, the buyer would always truthfully report the signal realization and (2) the buyer could not gain by jointly misreporting their type and purchasing more often than recommended. We apply a novel change of variable involving the threshold quality and the threshold type to show the desired result.

4 Posted Price Implementation

This section analyzes the role of price discrimination and information design. Sections 4.1 and 4.2 explore whether and when the seller could achieve the optimal profit by posting a non-discriminatory price for each product ex ante. Section 4.1 analyzes the case where the threshold type is constant in product quality. In this case, we show the seller can always achieve the goal under omniscient disclosure. If in addition the buyer's value has non-decreasing differences in product quality and buyer type, then the seller can also achieve the goal under private disclosure. Section 4.2 analyzes the case where the threshold type is decreasing in product quality. We first consider the case where the threshold type is strictly decreasing in product quality. In this case, we show the seller can achieve the goal under omniscient or private disclosure if and only if the buyer's value is identical for every quality-threshold type pair. Then we use an example to show that the intermediate case is indeterminate: if the threshold type is weakly but not strictly decreasing in product quality, then an ex-ante and non-discriminatory posted price mechanism cannot be optimal for all independent quality distributions, but may be optimal for some value/virtual functions *and* specific quality distributions. Section 4.3 explores the optimality of the no disclosure benchmark in the single product scenario of $m = 1$. Under omniscient or private disclosure, we show the seller cannot improve upon the no disclosure benchmark if the threshold type is constant in product quality. Under omniscient disclosure, we additionally show the seller can strictly improve upon the no disclosure benchmark when this condition fails. Taken together, this section sheds light on the applicability of the posted price mechanism in the presence of information design.

4.1 Posted Price Implementation: Common Threshold Type

The analysis in Section 3 shows that at the profit-maximizing allocation, the buyer is either allocated nothing or one unit of one of the products. Consequently, the total payment can be viewed as the price for the corresponding product. Nevertheless, the optimal price for each product generally depends either on the buyer's type (under the set of Blackwell least informative experiments) or the signal realization (under full disclosure). In many practical applications, however, the seller posts a non-discriminatory price for each product before conducting any experiments. We explore whether and when this pricing behavior can be consistent with profit maximization.

To answer this question, we consider how the threshold type $\theta^\star(q)$ (as defined in Equation 2) varies with the product quality q. Lemma 1 implies that $\theta^\star(q)$ must be non-increasing in q. This subsection analyzes the case where $\theta^\star(q)$ is constant in q, and the next subsection analyzes the case where $\theta^\star(q)$ is decreasing in q.

Before proceeding with the characterization, we define an experiment very similar to $E^*(\theta)$, but with no type-specific product. Specifically, let $S^* = \{s_1^*, s_2^*, \ldots, s_m^*\}$ be a signal space with m signal realizations. Further let $Q_i^* = \{\boldsymbol{q} \in Q : q_i \geq \max_{1\leq j<i} q_j, q_i > \max_{i<j\leq m} q_j\}$ be the collection of quality profiles in $Q := \times_{i=1}^m [\underline{q}_i, \bar{q}_i]$ where product i is of the highest realized quality. For any $\boldsymbol{q} \in Q_i^*$, let $\pi^*(s = s_i^*|\boldsymbol{q}) = 1$. This experiment is given by

$$E^* = (S^*, \pi^*)$$

The following proposition shows that when the threshold type is independent of product quality, the seller can always achieve the optimal profit by posting a non-discriminatory price for each product ex ante. Equivalently, price discrimination is of no additional value.

Proposition 2 (Posted Price Implementation: Common Threshold Type)**.** *Suppose Assumptions 1 through 4 hold and $\theta^\star(q) \equiv \theta^\star$ for any $q \in [\max_i \underline{q}_i, \max_i \bar{q}_i]$, then* *the seller can achieve the optimal profit, subject to (*IC_{om}*) and (IR), with menu* $(E^*, p_i = \mathbb{E}[v(q_i, \theta^\star)|s_i^*])$.

If in addition Assumption 5 holds and $v(q, \theta)$ *has non-decreasing differences in* (q, θ)*, then menu* $(E^*, p_i = \mathbb{E}[v(q_i, \theta^\star)|s_i^*])$ *achieves the same profit, when subject to (*IC_{pr}*) and (IR), as when subject to (*IC_{om}*) and (IR).*[12]

When the threshold type is independent of product quality, the buyer's threshold quality must be outside the range $(\max \underline{q}_i, \max \bar{q}_i)$. In other words, depending on their type, the buyer is either *always* allocated the product of highest realized quality, or *never* allocated any product at the optimum. Consequently, a posted price mechanism based on the threshold type can induce the profit-maximizing allocation. Given mutually independent product quality and complementarity between product quality and buyer type, the buyer's incentive constraints under private disclosure are respected as well.

Remark 5: There are two clarifications regarding the characterization in Proposition 2.

(1) The set of optimal experiments is not unique, and we present the optimal *non-discriminatory* experiment. In particular,

$$q^\star(\theta) = \begin{cases} \max_i \bar{q}_i & \text{if } \theta \le \theta^\star \\ \le \max_i \underline{q}_i & \text{if } \theta > \theta^\star \end{cases}$$

and the set of experiments $\{E^*(\theta)\}$ is discriminatory.

(2) The second part establishes the claim in Section 3.2 that the seller can achieve the same profit with private disclosure as with omniscient disclosure when the buyer's value has non-decreasing differences in product quality and buyer type and the threshold type is independent of product quality (case 2). Similar to Theorem 2, this result relies on mutually independent product quality.

[12] The prescribed menu is an indirect implementation of the optimal profit outcome, so the statement is slightly different from that in Theorem 2.

Figure 1 illustrates this case by plotting the threshold type as a function of product quality. Graphically, the level curve of $\varphi(q, \theta) = 0$ is a horizontal line at $\theta = \theta^\star$.

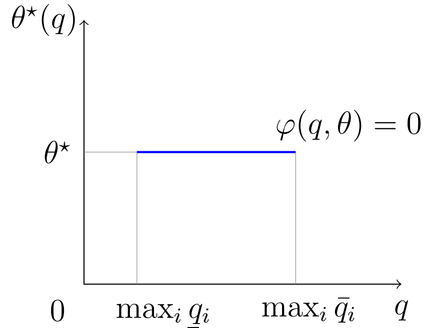


Figure 1: Common Threshold Type

Whereas the case of common threshold type is special, the following example shows it does capture some practically relevant value/virtual value functions.

Example 1 (Common Threshold Type)**.** *Consider a generalization of the numerical example in the introduction: $m = 2$, $c(q) = cq$ and $v(q, \theta) = q\theta$. Suppose $0 < c < 1$, $\theta \sim U[0, 1]$, q_1 and q_2 are two continuous and mutually independent random variables with $q_1 \sim F_1[\underline{q}_1, \bar{q}_1]$, $q_2 \sim F_2[\underline{q}_2, \bar{q}_2]$.*

$v(q, \theta)$ has strictly increasing differences in (q, θ) and $\theta^\star(q) = \frac{1+c}{2}$ for any q, so the conditions for Proposition 2 are satisfied. For both omniscient and private disclosures, the seller can achieve the optimal profit, subject to the buyer's incentive constraints, by conducting an experiment that reveals the identity of the product of higher quality and setting prices $p_i = \frac{1+c}{2} \cdot \mathbb{E}[q_i | q_i > q_{-i}]$, for $i = 1, 2$. Given this menu, the buyer optimally purchases the product of higher realized quality if their type is at least $\frac{1+c}{2}$. Otherwise, the buyer does not purchase any product under either signal realization.

More generally, the threshold type is independent of product quality if the buyer's value net of the seller's reservation value is multiplicatively separable, that is, $v(q, \theta) - c(q) = a(q) \cdot b(\theta)$. When $a(.), b(.) \geq 0$ are both strictly increasing, the condition of non-decreasing differences in product quality and buyer type is satisfied as well.

The next example illustrates that the condition of $v(.,.)$ having non-decreasing differences in (q, θ) is indispensable for incentive compatibility under private disclosure.

Example 2 (Necessity of Non-decreasing Differences of $v(q, \theta)$)**.** *Consider again the case of $m = 2$ and $\theta \sim U[0, 1]$. Let $c(q) = 0$ and*

$$v(q, \theta) = \begin{cases} q\theta & \text{if } 0 \le \theta \le \frac{3-q}{2} \\ (6-q)\theta - 2\theta^2 - \frac{(3-q)^2}{2} & \text{if } \frac{3-q}{2} < \theta \le 1 \end{cases}$$

$q_1 \equiv \frac{3}{2}$ and $q_2 \sim U[0, 2]$.

The threshold type $\theta^\star(q) = \frac{1}{2}$ for any $q \in [0, 2]$. Suppose the seller conducts an experiment that reveals the identity of the product of higher quality and sets prices $p_1 = \mathbb{E}[v(q_1, \frac{1}{2})|q_1 > q_2] = \frac{3}{4}$ and $p_2 = \mathbb{E}[v(q_2, \frac{1}{2})|q_2 > q_1] = \frac{7}{8}$. Since the experiment is non-discriminatory, the only set of relevant constraints under omniscient disclosure is (IR). The buyer would indeed purchase the offered product under either signal realization if $\theta > \frac{1}{2}$. Under private disclosure, however, when the experiment reveals $q_2 > q_1$, for a buyer with type $\theta = 1$, they would optimally purchase product 1 (which yields a surplus of $\frac{5}{8}$) instead of product 2 (which yields a surplus of $\frac{7}{12}$). The same holds for any θ close to 1, so (IC_{pr}) is violated. In other words, this menu cannot achieve the same profit with private disclosure as with omniscient disclosure.[13]

For the menu in Proposition 2, a buyer with the threshold type would never misreport the signal realization. If $v(.,.)$ has non-decreasing differences in (q, θ), then Lemma 3 shows that nor would any buyer with type $\theta' > \theta$. The assumption of common threshold type only guarantees that $v(.,.)$ has non-decreasing differences in (q, θ) around the threshold type, but the *local* condition may not hold *globally*. In Example 2, $v(.,.)$ has decreasing differences in (q, θ) when they are both close to their upper bounds. Intuitively, this implies that given a high product quality, a high type buyer may have a lower marginal value for quality compared with the threshold type buyer, which explains why they may opt for the lower expected quality product. As discussed in Section 3.2, there is no such deviation when there is only one product,

[13]This example can adapted such that q_1 is non-degenerate.

which showcases the additional complication in guaranteeing incentive compatibility with multiple products under private disclosure.

4.2 Posted Price Implementation: Decreasing Threshold Type

In this subsection, we turn to the case where $\theta^\star(q)$ is decreasing in q. Let $q_E = \max\{\max_i \underline{q}_i, q^\star(\bar{\theta})\}$, then $[q_E, \max_i \bar{q}_i]$ is the range of highest product quality realization relevant for the profit-maximizing allocation and $[q_E, q^\star(\underline{\theta})] \subset [q_E, \max_i \bar{q}_i]$ is the range where $\theta^\star(q)$ may vary with q. We start by considering the case where $\theta^\star(q)$ is strictly decreasing on $[q_E, q^\star(\underline{\theta})]$.

Before preceding with the characterization, we describe the experiment that involves full disclosure. Let $S = Q$ be the signal space. Moreover, for any $\boldsymbol{q} \in Q$ and $\boldsymbol{q}' \neq \boldsymbol{q}$, let $\pi^{\text{full}}(s = \boldsymbol{q}|\boldsymbol{q}) = 1$ and $\pi^{\text{full}}(s = \boldsymbol{q}'|\boldsymbol{q}) = 0$. This experiment is given by

$$E^{\text{full}} = (Q, \pi^{\text{full}})$$

The following proposition fully characterizes the condition under which price discrimination is of no additional value when $\theta^\star(.)$ is strictly decreasing.

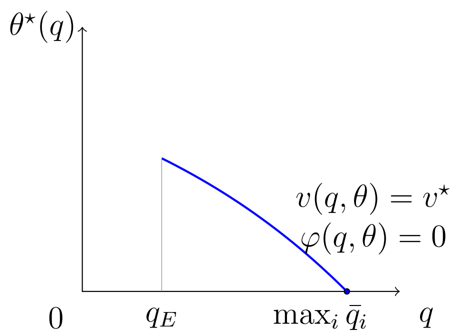
Proposition 3 (Posted Price Implementation: Strictly Decreasing Threshold Type)**.** *Suppose Assumptions 1 through 4 hold and $\theta^\star(q)$ is strictly decreasing in q on $[q_E, q^\star(\underline{\theta})]$, then the seller can achieve the optimal profit, subject to (IC_{om}) and (IR), with a set of experiments $\{\tilde{E}(\theta)\}$ and an ex-ante posted price mechanism $(p_i)_{i=1}^m$ if and only if $v(q, \theta^\star(q)) \equiv v^\star$, for any $q \in [q_E, \max_i \bar{q}_i]$.*

When this condition holds, one optimal menu is $\left(E^{full}, p_i = v^\star\right)$. Moreover, this menu achieves the same profit, when subject to (IC_{pr}) and (IR), as when subject to (IC_{om}) and (IR).

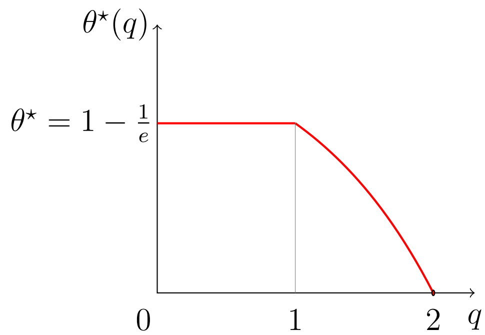
In other words, given strictly decreasing threshold type, price discrimination is not beneficial to the seller if and only if the buyer's value is identical for every quality-threshold type pair. To understand this result, notice if an ex-ante and non-discriminatory posted price mechanism is optimal, then the buyer's payment for any product cannot depend on their type in the optimal menu involving the set of Blackwell least informative experiments. For the incentive constraints to be respected, the price for any product i such that $\bar{q}_i > q_E$ must be identical, which implies that the buyer's value must be identical for every quality-threshold type pair.

Remark 6: Similar to the characterization in Proposition 2, the set of optimal experiments is not unique, and we present the optimal *non-discriminatory* experiment.

The left panel of Figure 2 illustrates the special case in Proposition 3 by plotting the threshold type as a function of product quality. Graphically, the level curve of $\varphi(q,\theta) = 0$ coincides with a level curve of $v(q,\theta)$.



(a) $v(q,\theta^\star(q)) = v^\star$



(b) Example 4

Figure 2: Decreasing Threshold Type

The case of strictly decreasing threshold type captures almost all value/virtual value functions, but the case of common value for every quality-threshold type pair is very special. The following example offers one possibility using a special class of additively separable valuations.

Example 3 (Common Value for Quality-Threshold Type Pair)**.** *Consider again the case of $m = 2$ and $\theta \sim U[0,1]$. Let $c(q) = 0$ and*

$$v(q,\theta) = \begin{cases} \ln(\frac{1}{1-\theta}) + q & \text{if } 0 \leq \theta \leq 1 - \frac{1}{e} \\ e\theta + q + 2 - e & \text{if } 1 - \frac{1}{e} < \theta \leq 1 \end{cases}$$

Each q_i can be an arbitrary random variable with support on $[\underline{q}_i, \bar{q}_i]$ such that $0 \leq \underline{q}_i <$

$\bar{q}_i \leq 1$.

The threshold type $\theta^\star(q) = 1 - e^{q-1}$ and $v(q, \theta^\star(q)) = 1$ for any $q \in [0, 1]$, so the conditions in Proposition 3 are satisfied. For both omniscient and private disclosures, the seller can achieve the optimal profit, subject to the buyer's incentive constraints, by conducting an experiment that reveals the exact quality of the two products and setting prices $p_i = 1$, for $i = 1, 2$. Given this menu, the buyer optimally purchases the product of higher realized quality if $q \geq q^\star(\theta) = \max\{1 - \ln(\frac{1}{1-\theta}), 0\}$.

A slightly more general buyer's value function can be constructed by modifying $v(q, \theta) = b\ln(\frac{1}{1-\theta}) + a(q)$ to make it well-defined at $\theta = 1$ for any $b > 0$ and $a(.) < b$ non-decreasing, as in Example 3.

Finally, we consider the case where $\theta^\star(q)$ is weakly but not strictly decreasing on $[q_E, q^\star(\underline{\theta})]$. The characterizations in Propositions 2 and 3 are distribution-free for quality, that is, given $\max_i \bar{q}_i$ and $\max_i \underline{q}_i$, we are able to identify conditions on $v(q, \theta)$ and $\varphi(q, \theta)$ such that price discrimination is not beneficial for *any* joint quality distribution. The following example shows this is not possible when $\theta^\star(q)$ is constant in some range, but strictly decreasing in some other range on $[q_E, q^\star(\underline{\theta})]$. Specifically, there exist value/virtual value functions such that price discrimination is not beneficial for some joint quality distributions, but beneficial for some others.

Example 4 (Possible Indeterminacy with Weakly Decreasing Threshold Type)**.** *Consider again the case of $m = 2$ and $\theta \sim U[0, 1]$. Let $c(q) = 0$ and*

$$v(q, \theta) = \begin{cases} q\ln(\frac{1}{1-\theta}) & \text{if } 0 \leq \theta \leq 1 - \frac{1}{e} \text{ and } 0 \leq q \leq 1 \\ \ln(\frac{1}{1-\theta}) + q - 1 & \text{if } 0 \leq \theta \leq 1 - \frac{1}{e} \text{ and } 1 < q \leq 2 \\ q + qe\left[\theta - (1 - \frac{1}{e})\right] & \text{if } 1 - \frac{1}{e} < \theta \leq 1 \text{ and } 0 \leq q \leq 1 \\ q + e\left[\theta - (1 - \frac{1}{e})\right] & \text{if } 1 - \frac{1}{e} < \theta \leq 1 \text{ and } 1 < q \leq 2 \end{cases}$$

- *Scenario 1: $q_1 \sim U[0, 2]$ and $q_2 \sim U[\underline{q}_2, \bar{q}_2]$, where $0 < \underline{q}_2 < 1 < \bar{q}_2 < 2$.*
- *Scenario 2: $q_2 \sim U[0, 2]$ and $q_2 \equiv \frac{3}{4}$.*

The threshold type as function of product quality is plotted in the right panel of

Figure 2 and is given by

$$\theta^\star(q) = \begin{cases} 1 - \frac{1}{e} & \text{if } 0 \leq q \leq 1 \\ 1 - e^{q-2} & \text{if } 1 < q \leq 2 \end{cases}$$

For both omniscient and private disclosures, the seller can achieve the optimal profit, subject to the buyer's incentive constraints, with an ex-ante and non-discriminatory posted price mechanism in Scenario 2, but not in Scenario 1.

An analogous argument to the proof of Lemma 2 shows it is without loss to restrict to the set of experiments $\{E^*(\theta)\}$. Notice $v(q, \theta^\star(q)) = 1$ for $q > 1$, so an analogous argument to the proof of Proposition 3 shows the only possible posted prices in Scenario 1 are $p_1 = p_2 = 1$. Since $q^\star(1 - \frac{1}{e}) = 0$ and $v(q, 1 - \frac{1}{e}) < 1$ for $0 < q < 1$, buyer type $\theta^\star = 1 - \frac{1}{e}$ would prefer the menu with experiment $E^*(1 - \frac{1}{e} - \varepsilon)$ for $\varepsilon > 0$ small, which contradicts with profit maximization.

In Scenario 2, however, the seller can achieve the optimal profit by offering the set of experiments $\{E^*(\theta)\}$ and setting prices $p_1 = 1$ and $p_2 = \frac{17}{24}$. Intuitively, since $\bar{q}_2 < 1$, the buyer's local incentive constraints are not sufficient to pin down p_2. By setting $p_2 < v(q_2, 1 - \frac{1}{e})$, the seller can induce buyer type $\theta^\star = 1 - \frac{1}{e}$ to take the menu designed for their type. This is only possible when $m > 1$. For $m = 1$, the buyer can gain information rent only from one product, which is uniquely pinned down by the envelope formula (recall item 2 of Remark 2).[14]

Remark 7: An almost identical argument to the proof of Proposition 3 and Example 4 can establish the following two statements.

(1) For $m = 1$, Propositions 2 and 3 characterize all the possibilities where the seller can achieve the optimal profit by posting a non-discriminatory price ex ante.

(2) For $m > 1$, if $\theta^\star(q)$ is weakly but not strictly decreasing on $[q_E, q^\star(\underline{\theta})]$, then for any $v(q,\theta)$ and $\varphi(q,\theta)$, there exist quality distributions that are mutually

[14] Similar to Example 2, this example can be adapted such that q_2 is non-degenerate.

independent such that the seller *cannot* achieve the optimal profit by posting a non-discriminatory price ex ante.

The following table summarizes the results in Sections 4.2 and 4.3.

How $\theta^\star(q)$ varies with q	Whether price discrimination is beneficial
$\theta^\star(q) \equiv \theta^\star$	Omniscient disclosure: No for any quality distributions. Private disclosure: No for any independent quality distributions if $v(q,\theta)$ has non-decreasing differences.
$\theta^\star(q)$ is strictly decreasing	Omniscient or private disclosure: No for any quality distributions if $v(q,\theta^\star(q)) \equiv v^\star$; Yes for any quality distributions otherwise.
$\theta^\star(q)$ is weakly decreasing	Omniscient or private disclosure: Yes in general; No for specific value/virtual value functions *and* quality distributions (only if $m \geq 2$).

4.3 Single-product Scenario

Our analysis suggests that no disclosure is hardly ever optimal when there are multiple products. Intuitively, there are two roles of information design: first, information disclosure can induce a more efficient allocation and create a larger surplus for the seller to extract from; second, information design can facilitate the seller to better screen the buyer and reap a larger share of the surplus. With multiple products, the first effect is always at work, even when the seller always wants to sell to the same buyer types irrespective of the quality realization. Based on this observation, we focus on the single-product scenario of $m = 1$, and analyze whether and when no disclosure is consistent with profit maximization.

For notational simplicity, we write the product quality as $q \sim G[\underline{q}, \bar{q}]$. Before presenting the characterization, we first define the optimal no disclosure menu. Let $S^{\text{no}} = \{s_0\}$ be a singleton set. For any $\boldsymbol{q} \in Q$, let $\pi^{\text{no}}(s = s_0|\boldsymbol{q}) = 1$. Then no disclosure is given by the experiment

$$E^{\text{no}} = (S^{\text{no}}, \pi^{\text{no}})$$

With a slight abuse of notation, let $\theta^\star$ be the threshold type with no disclosure, that is,

$$\theta^\star = \begin{cases} \bar{\theta} & \text{if } \mathbb{E}[\varphi(q,\theta)] \leq 0 \text{ for all } q \in [\underline{q}, \bar{q}] \\ \inf\{\theta \in [\underline{\theta}, \bar{\theta}] : \mathbb{E}[\varphi(q,\theta)] > 0\} & \text{otherwise} \end{cases}$$

The optimal no disclosure menu is $(E^{\mathrm{no}}, p = \mathbb{E}[v(q, \theta^\star)])$. The following proposition characterizes the circumstances under which no disclosure can be consistent with profit maximization for $m = 1$.

Proposition 4 (Optimality of No Disclosure for $m = 1$)**.** *Suppose Assumptions 1, 2, 4 hold and $m = 1$, then menu $(E^{no}, p = \mathbb{E}[v(q, \theta^\star)])$ maximizes the seller's expected profit, subject to (IC$_{om}$) and (IR), if and only if $\theta^\star(q) = \theta^\star$, for any $q \in [\underline{q}, \bar{q}]$. Moreover, this menu achieves the same profit, when subject to (IC$_{pr}$) and (IR), as when subject to (IC$_{om}$) and (IR).*

Given our analysis, Proposition 4 is immediate. By Theorem 1, the seller can maximize their expected profit, subject to (IC$_{\mathrm{om}}$) and (IR), if the expected utility of the lowest type is 0 and the buyer is allocated the product if the quality realization is above their threshold quality. With a constant threshold type, both conditions are satisfied when the buyer chooses optimally in the no disclosure benchmark. When the threshold type is not constant, the threshold quality must be different for a continuum of buyer types, so the buyer's optimal choice must be different from the profit-maximizing allocation with a positive probability.

Remark 8: Proposition 4 shows that whether information design is beneficial to a single-product monopolist crucially relies on whether the threshold type depends on the product quality. For instance, Smolin (2023) shows that no disclosure is optimal when $v(q, \theta) = q\theta$ and $c(q) = 0$, whereas Wei and Green (2023) show that no disclosure is not optimal when $c(q) = c > 0$. Our analysis suggests that the disparity is not so much because it is always efficient to sell the product in the first case, nor is it so much because the seller has a cost in the second case. The essential difference is that the threshold type is independent of product quality in the first but not the second case. Intuitively, Propositions 2 and 3 suggest that the seller has additional incentives to screen the buyer in the second case. On the other hand, Example 1 shows that if the seller's cost is also linear in quality, that is, $c(q) = cq$, then no disclosure remains optimal.

5 Concluding Remarks

In this paper, we analyzed the profit-maximizing menu of experiments and selling mechanisms under both omniscient and private disclosures for a monopolist seller with multiple products of distinct quality. In addition to the characterization of the optimal menu, there are two take-aways: first, information disclosure can almost always increase the seller's expected profit when a more efficient allocation can be induced; second, price discrimination, when combined with information design, can improve the seller's expected profit only when the minimum type of buyer to which the seller wants to relinquish a product depends on the product's quality realization.

For future research, we view three directions as particularly worthy of investigation: first, it is challenging but important to provide a complete characterization of the monopolist's profit-maximizing menu under private disclosure. Such a characterization would offer more insight into the seller's trade-off between creating a larger surplus through a more informative experiment and leaving less information rent through a less informative experiment for some buyer types; second, if we view the single buyer as a collection of buyers of measure one, then consumption externalities may be an important consideration in some practical applications; third, competition among sellers can also play an important role. To be concrete, if every seller produces an identical set of products, then the insight of Kamenica and Gentzkow (2017) suggests that more information is likely to be revealed at the optimum and our analysis can be generalized. On the other hand, if each seller produces a distinct set of products and can conduct experiments only with respect to the quality of their own products, then how competition would affect the optimal disclosure policy is far from obvious, and even less can be said about the interaction with optimal pricing.[15]

[15] An interesting special case is where there are two asymmetric sellers, one with products of deterministic quality and the other with products of uncertain quality. We thank Yingni Guo for this suggestion.

Appendix

In this appendix, we provide the proofs omitted from the main text.

Proof of Lemma 1: Suppose not, then $\exists \theta < \theta'$ such that $q^\star(\theta) < q^\star(\theta')$. By the definition of $q^\star(\theta')$, we must have for any $q \in (q^\star(\theta), q^\star(\theta'))$, $\varphi(q, \theta') \leq 0$. By Assumption 2, we have

$$\varphi(q, \theta) < \varphi(q, \theta') \leq 0 \tag{A1}$$

On the other hand, $q > q^\star(\theta)$, so by Assumption 2 again, we have for any $\delta \in (0, q - q^\star(\theta))$,

$$\varphi(q, \theta) > \varphi(q^\star(\theta) + \delta, \theta) > 0,$$

which contradicts with inequality (A1). $\square$

Proof of Theorem 1: We prove the lemma in three steps: first, we show a necessary condition of the set of (IC_{om}) constraints is the envelope formula; second, we solve for a candidate optimal menu of experiments and mechanisms; third, we show the sets of incentive constraints, (IC_{om}) and (IR), are indeed satisfied at the proposed menu of experiments and mechanisms.

Step 1: Let $V(\theta'; \theta) = \int_{s \in S(\theta')} U(\theta', s; \theta)\pi(\theta', s)ds$ be the expected utility of a type θ buyer when they report and act as being type θ' and let $V(\theta) := V(\theta; \theta)$ be the expected utility when they report truthfully. Then a necessary condition of the set of (IC_{om}) constraints is the following envelope formula:

$$V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \left[\int_{s \in S(\eta)} \left(\sum_{i=1}^{m} x_i(\eta, s)\mathbb{E}[v_\eta(q_i, \eta) | s, E(\eta)] \right) \pi(\eta, s)ds \right] d\eta \tag{A2}$$

Let $\delta(\theta'; \theta) = V(\theta'; \theta) - V(\theta)$ be the expected utility gain of a type θ buyer when they report and act as being type θ'. Notice $\int_{s \in S(\theta')} \max\{U(\theta', s; \theta), 0\}\pi(\theta', s)ds \geq \int_{s \in S(\theta')} U(\theta', s; \theta)\pi(\theta', s)ds$, so the set of ($\text{IC}_{\text{om}}$) constraints implies $\delta(\theta'; \theta) \leq 0 = \delta(\theta'; \theta')$. For any $\theta_1, \theta_2 \in [\underline{\theta}, \bar{\theta}]$, we have $|V(\theta_1) - V(\theta_2)| \leq |V(\theta_1) - V(\theta_1; \theta_2)| = |\int_{s \in S(\theta_1)} [U(\theta_1, s) - U(\theta_1, s; \theta_2)]\pi(\theta_1, s)ds| \leq \max_{q,\theta} |v_\theta(q, \theta)||\theta_1 - \theta_2|$, where the last inequality is by Assumption 3 of unit demand. In other words, $V(.)$ is Lipschitz continuous and, in particular, absolutely continuous and differentiable almost everywhere. Consequently, $\frac{\partial \delta(\theta';\theta)}{\partial \theta}|_{\theta=\theta'} = \frac{\partial V(\theta';\theta)}{\partial \theta}|_{\theta=\theta'} - V'(\theta') = 0$ and

$$V'(\theta) = \int_{s \in S(\theta)} \left(\sum_{i=1}^{m} x_i(\theta, s) \mathbb{E}[v_\theta(q_i, \theta)|s, E(\theta)] \right) \pi(\theta, s) ds$$

By the fundamental theorem of calculus, we obtain envelope formula (A2).

Step 2: The seller's expected profit is given by

$$\text{EP} = \mathbb{E}_\theta \left[\int_{s \in S(\theta)} t(\theta, s) \pi(\theta, s) ds \right] - \mathbb{E}_\theta \left[\int_{s \in S(\theta)} \left(\sum_{i=1}^{m} x_i(\theta, s) \mathbb{E}[c(q_i)|s, E(\theta)] \right) \pi(\theta, s) ds \right]$$

Notice that

$$\int_{s \in S(\theta)} t(\theta, s) \pi(\theta, s) ds = \int_{s \in S(\theta)} \left(\sum_{i=1}^{m} x_i(\theta, s) \mathbb{E}[v(q_i, \theta)|s, E(\theta)] \right) \pi(\theta, s) ds - V(\theta)$$

Together with the envelope formula, we have

$$\text{EP} = \mathbb{E}_\theta \left[\int_{s \in S(\theta)} \left(\sum_{i=1}^{m} x_i(\theta, s) \mathbb{E} \left[v(q_i, \theta) - c(q_i) - \frac{1 - F(\theta)}{f(\theta)} v_\theta(q_i, \theta) | s, E(\theta) \right] \right) \pi(\theta, s) ds \right] - V(\underline{\theta})$$

Recall the buyer's virtual value $\varphi(q, \theta) := v(q, \theta) - c(q) - \frac{1-F(\theta)}{f(\theta)} v_\theta(q, \theta)$. It follows that the seller's expected profit can be simplified as

$$\text{EP} = \mathbb{E}_\theta \left[\int_{s \in S(\theta)} \left(\sum_{i=1}^{m} x_i(\theta, s) \mathbb{E}[\varphi(q_i, \theta)|s, E(\theta)] \right) \pi(\theta, s) ds \right] - V(\underline{\theta}) \tag{A3}$$

The set of (IR) constraints implies $V(\underline{\theta}) \geq 0$, so in order to maximize Equation (A3), the seller chooses $U(\underline{\theta}, s) = 0$, for any $s \in S(\underline{\theta})$, so that $V(\underline{\theta}) = 0$. Moreover, by Assumption 2, $\varphi(., \theta)$ is strictly increasing in q, for any θ and $q \geq q^\star(\theta)$. By choosing $E(\theta) = E^*(\theta)$ and $x_i^*(\theta, s_i^*(\theta)) = 1$, the seller can maximize their expected profit for every $\boldsymbol{q} \in Q$.

Step 3: It remains to show the two sets of incentive constraints, (IC_{om}) and (IR), are both respected at the proposed menu of experiments and mechanisms. First, for any $\theta \in [\underline{\theta}, \bar{\theta}]$ and $i = 1, 2, \ldots, m$,

$$U(\theta, s_0^*(\theta)) = 0 \quad \text{and} \quad U(\theta, s_i^*(\theta)) = \frac{V_i(\theta)}{\pi^*(\theta, s_i^*(\theta))} \geq 0$$

In other words, the set of (IR) constraints is respected.

Next, for the set of (IC_{om}) constraints, we need to verify that for any $\theta, \theta' \in [\underline{\theta}, \bar{\theta}]$,

$$\sum_{i=1}^{m} V_i(\theta) \geq \sum_{i=1}^{m} \left(\max \left\{ \mathbb{E}[v(q_i, \theta) | s_i^*(\theta')] - \mathbb{E}[v(q_i, \theta') | s_i^*(\theta')] + \frac{V_i(\theta')}{\pi^*(\theta', s_i^*(\theta'))}, 0 \right\} \right) \pi^*(\theta', s_i^*(\theta'))$$

Since $V_i(\theta) \geq 0$, for any $\theta \in [\underline{\theta}, \bar{\theta}]$, it suffices to show

$$V_i(\theta) \geq \left(\mathbb{E}[v(q_i, \theta) | s_i^*(\theta')] - \mathbb{E}[v(q_i, \theta') | s_i^*(\theta')] \right) \pi^*(\theta', s_i^*(\theta')) + V_i(\theta') \tag{A4}$$

Let

$$A_i(\theta'; \theta) = V_i(\theta) - V_i(\theta')$$

$$B_i(\theta'; \theta) = \left(\mathbb{E}[v(q_i, \theta) | s_i^*(\theta')] - \mathbb{E}[v(q_i, \theta') | s_i^*(\theta')] \right) \pi^*(\theta', s_i^*(\theta'))$$

Then Inequality (A4) is equivalent to $A_i(\theta'; \theta) \geq B_i(\theta'; \theta)$, for any $\theta, \theta' \in [\underline{\theta}, \bar{\theta}]$.

Let $\tilde{Q}_i^*(\eta) = \{\boldsymbol{q} \in Q : s = s_i^*(\eta) \text{ given } \boldsymbol{q}\}$ be the collection of actual quality profiles where signal $s_i^*(\eta)$ is sent. Lemma 1 implies that $\tilde{Q}_i^*(\eta) \subset \tilde{Q}_i^*(\eta')$, for any $\eta \leq \eta'$ and $i = 1, 2, \ldots, m$. Consequently, by the Fubini's theorem (i.e., changing the order of the integral) and Assumption 1, we have

$$\begin{aligned} B_i(\theta'; \theta) &= \int_{\boldsymbol{q} \in \tilde{Q}_i^*(\theta')} \left(v(q_i, \theta) - v(q_i, \theta') \right) dG(\boldsymbol{q}) \\ &= \int_{\boldsymbol{q} \in \tilde{Q}_i^*(\theta')} \left(\int_{\theta'}^{\theta} v_\eta(q_i, \eta) d\eta \right) dG(\boldsymbol{q}) \\ &= \int_{\theta'}^{\theta} \left(\int_{\boldsymbol{q} \in \tilde{Q}_i^*(\theta')} v_\eta(q_i, \eta) dG(\boldsymbol{q}) \right) d\eta \end{aligned}$$

$$
\begin{aligned}
&\le \int_{\theta'}^{\theta}\left(\int_{\boldsymbol{q}\in\tilde{Q}_i^*(\eta)} v_\eta(q_i,\eta)dG(\boldsymbol{q})\right)d\eta \\
&= A_i(\theta';\theta)
\end{aligned}
$$

We conclude that the two sets of incentive constraints, (IC_{om}) and (IR), are both respected at the proposed menu of experiments and mechanisms. □

Proof of Proposition 1: Let $\Theta^0 = \{\theta \in [\underline{\theta}, \bar{\theta}] : \tilde{E}(\theta) \not\succ E^*(\theta)\}$ be the collection of buyer types where $\tilde{E}(\theta)$ is not sufficient/Blackwell more informative than $E^*(\theta)$ (Blackwell, 1953). Suppose by contradiction that $\mathbb{P}(\theta \in \Theta^0) > 0$.

Now consider any $\theta \in \Theta^0$. Since $\tilde{E}(\theta) \not\succ E^*(\theta)$, there exists a signal $s^0 \in \tilde{S}(\theta)$ and two collections of quality profiles $Q_i^0 \subset Q_i^*(\theta)$ and $Q_j^0 \subset Q_j^*(\theta)$, with $i, j \in \{0, 1, 2, \ldots, m\}$ and $\mathbb{P}(Q_i^0), \mathbb{P}(Q_j^0) > 0$, such that $\tilde{\pi}(\theta)(s^0|\boldsymbol{q}^i), \tilde{\pi}(\theta)(s^0|\boldsymbol{q}^j) > 0$, for any quality profile $\boldsymbol{q}^i \in Q_i^0$ and $\boldsymbol{q}^j \in Q_j^0$. (Otherwise for any $\tilde{s}(\theta) \in \tilde{S}(\theta)$, there exists $l \in \{0, 1, 2, \ldots, m\}$ such that $\int_{\boldsymbol{q}\in Q_l^*(\theta)} \tilde{\pi}(\theta, \tilde{s}(\theta)|\boldsymbol{q})d\boldsymbol{q} = 1$ and $\tilde{E}(\theta)$ would be sufficient for $E^*(\theta)$.) Consequently, for a positive measure of quality realization $\boldsymbol{q} \in Q_i^0 \cup Q_j^0$, the allocation associated with experiment $\tilde{E}(\theta)$, $\tilde{\boldsymbol{x}}(\theta, \boldsymbol{q}) \neq \boldsymbol{x}^*(\theta, \boldsymbol{q})$. On the other hand, by the proof of Theorem 1, menu $(\{E^*(\theta)\}, \{\boldsymbol{x}^*(\theta, s_i^*(\theta)), t^*(\theta, s_i^*(\theta))\})$ maximizes the seller's profit for almost all $\theta \in [\underline{\theta}, \bar{\theta}]$ and $\boldsymbol{q} \in Q$, subject to (IC_{om}) and (IR). We thus have,

$$
\int_{\tilde{s}\in\tilde{S}(\theta)} \tilde{t}(\theta, \tilde{s})\tilde{\pi}(\theta, \tilde{s})ds < \int_{s\in S^*(\theta)} t^*(\theta, s)\pi^*(\theta, s)ds, \tag{A5}
$$

for almost all $\theta \in \Theta^0$.

$$
\int_{\tilde{s}\in\tilde{S}(\theta)} \tilde{t}(\theta, \tilde{s})\tilde{\pi}(\theta, \tilde{s})ds \le \int_{s\in S^*(\theta)} t^*(\theta, s)\pi^*(\theta, s)ds, \tag{A6}
$$

for all $\theta \in [\underline{\theta}, \bar{\theta}] \backslash \Theta^0$.

Let $\text{EP}(\tilde{E}(\theta))$ be the expected profit associated with $\{\tilde{E}(\theta)\}$ and the corresponding incentive-compatible selling mechanism and let $\text{EP}(E^*(\theta))$ be the expected profit characterized in Theorem 1. Inequalities (A5) and (A6) together imply that $\text{EP}(\tilde{E}(\theta)) < \text{EP}(E^*(\theta))$, which contradicts with the assumption of profit maximization. □

Proof of Lemma 2: By Proposition 1, $\tilde{E}(\theta)$ is sufficient/Blackwell more informative than $E^*(\theta)$, for almost all θ. Since $E^*(\theta)$ is standard, the probability measure induced by $\tilde{\pi}(\theta)$ is a stochastic transformation of the probability measure induced by $\pi^*(\theta)$ (Blackwell, 1953). For any $\boldsymbol{q} \in Q_i^*(\theta)$, we know $\pi^*(\theta, s|\boldsymbol{q}) = 0$, for any $s \neq s_i^*(\theta)$. Consequently, we can partition the signal space $\tilde{S}(\theta)$ into $m+1$ sets $\{\tilde{S}_i(\theta)\}_{i=0}^m$ such that for any $\boldsymbol{q} \in Q_i^*(\theta)$, $\tilde{\pi}(\theta, s|\boldsymbol{q}) = 0$, for any $s \in \tilde{S}(\theta) \backslash \tilde{S}_i(\theta)$.

Let $R_i(\theta, s) := \frac{\tilde{\pi}(\theta,s)}{\pi^*(\theta, s_i^*(\theta))}$. Then $R_i(\theta, s)$ induces a probability measure on $\tilde{S}_i(\theta)$, that is, $\int_{s\in\tilde{S}_i(\theta)} R_i(\theta, s)ds = 1$, for almost all θ and any $i = 0, 1, \dots, m$.

Let $t(\theta, s_i^*(\theta)) := \int_{s\in\tilde{S}_i(\theta)} \tilde{t}(\theta, s)R_i(\theta, s)ds$. By the proof of Theorem 1, as long as $V(\underline{\theta}) = 0$ and the incentive constraints are respected, menu $(\{E^*(\theta)\}, \{\boldsymbol{x}^*(\theta, s_i^*(\theta)), t(\theta, s_i^*(\theta))\})$ would maximize the seller's expected profit, subject to (IC_{om}) and (IR). Since (IC_{pr}) implies (IC_{om}), if we could show that menu $(\{E^*(\theta)\}, \{\boldsymbol{x}^*(\theta, s(\theta)), t(\theta, s(\theta))\})$ respects (IC_{pr}) and (IR) and $V(\underline{\theta}) = 0$, then we know the menu would also maximize the seller's expected profit, subject to the two sets of incentive constraints.

First, since menu $(\{\tilde{E}(\theta) = (\tilde{S}(\theta), \tilde{\pi}(\theta))\}, \{\tilde{\boldsymbol{x}}(\theta, \tilde{s}(\theta)), \tilde{t}(\theta, \tilde{s}(\theta))\})$ maximizes the seller's expected profit, subject to (IC_{om}) and (IR), by the proof of Theorem 1, $\tilde{x}(\theta, \boldsymbol{q}) = \boldsymbol{x}^*(\theta, \boldsymbol{q})$ for almost all $\theta \in [\underline{\theta}, \bar{\theta}]$ and $\boldsymbol{q} \in Q$. Together with the definition of $t(\theta, s_i^*(\theta))$, we know $V(\underline{\theta}) = 0$.

Next, we consider the set of (IR) constraints. Again by profit maximization of menu $(\{\tilde{E}(\theta) = (\tilde{S}(\theta), \tilde{\pi}(\theta))\}, \{\tilde{\boldsymbol{x}}(\theta, \tilde{s}(\theta)), \tilde{t}(\theta, \tilde{s}(\theta))\})$, $\tilde{\boldsymbol{x}}(\theta, s) = \boldsymbol{0}$, for any $s \in \tilde{S}_0(\theta)$. Together with (IR), we must have $\tilde{t}(\theta, s) = 0$, for any $s \in \tilde{S}_0(\theta)$, which implies $t(\theta, s_0^*(\theta)) = 0$. We thus have for any $\theta \in [\underline{\theta}, \bar{\theta}]$ and $i = 1, 2, \dots, m$,

$$U(\theta, s_0^*(\theta)) = -t(\theta, s_0^*(\theta)) = 0 \quad \text{and}$$

$$U(\theta, s_i^*(\theta)) = \int_{s\in\tilde{S}_i(\theta)} \left(\mathbb{E}[v(q_i, \theta)|s] - \tilde{t}(\theta, s)\right) R_i(\theta, s)ds = \int_{s\in\tilde{S}_i(\theta)} U(\theta, s)R_i(\theta, s)ds \geq 0$$

Equivalently, menu $(\{E^*(\theta)\}, \{\boldsymbol{x}^*(\theta, s_i^*(\theta)), t(\theta, s_i^*(\theta))\})$ respects (IR).

Finally, we consider the set of (IC_{pr}) constraints. For any $\theta, \theta' \in [\underline{\theta}, \bar{\theta}]$, we have

$$\begin{aligned}
& \sum_{i=0}^{m} \max_{0 \le j \le m} \left\{ U(\theta', s_j^*(\theta'); \theta, s_i^*(\theta')), 0 \right\} \pi(\theta', s_i^*(\theta')) \\
= & \sum_{i=0}^{m} \max_{1 \le j \le m} \left\{ \mathbb{E}[v(q_j, \theta) | s_i^*(\theta')] - t(\theta', s_j^*(\theta')), 0 \right\} \pi(\theta', s_i^*(\theta')) \\
= & \sum_{i=0}^{m} \max_{1 \le j \le m} \left\{ \int_{s \in \tilde{S}_i(\theta')} \mathbb{E}[v(q_j, \theta) | s] R_i(s, \theta') ds - \int_{s' \in \tilde{S}_j(\theta')} t(\theta', s') R_j(s', \theta') ds', 0 \right\} \pi(\theta', s_i^*(\theta')) \\
\le & \sum_{i=0}^{m} \max_{1 \le j \le m} \left\{ \int_{s \in \tilde{S}_i(\theta')} \left(\mathbb{E}[v(q_j, \theta) | s] - \inf_{s' \in \tilde{S}_j(\theta')} t(\theta', s') \right) R_i(s, \theta') ds, 0 \right\} \pi(\theta', s_i^*(\theta')) \\
= & \sum_{i=0}^{m} \max_{1 \le j \le m} \left\{ \int_{s \in \tilde{S}_i(\theta')} \sup_{s' \in \tilde{S}_j(\theta')} U(\theta', s'; \theta, s) R_i(s, \theta') ds, 0 \right\} \pi(\theta', s_i^*(\theta')) \\
\le & \sum_{i=0}^{m} \int_{s \in \tilde{S}_i(\theta')} \sup_{s'} \left\{ U(\theta', s'; \theta, s), 0 \right\} \pi(\theta', s) ds \\
= & \int_{s \in \tilde{S}(\theta')} \sup_{s'} \left\{ U(\theta', s'; \theta, s), 0 \right\} \pi(\theta', s) ds \\
\le & \int_{s \in \tilde{S}(\theta)} U(\theta, s) \pi(\theta, s) ds \\
= & \sum_{i=0}^{m} U(\theta, s_i^*(\theta)) \pi(\theta, s_i^*(\theta))
\end{aligned}$$

We conclude that menu $(\{E^*(\theta)\}, \{\boldsymbol{x}^*(\theta, s_i^*(\theta)), t(\theta, s_i^*(\theta))\})$ also respects (IC_{pr}). $\square$

Proof of Lemma 3: Since menu $(\{E^*(\theta)\}, \{\boldsymbol{x}(\theta, s_i^*(\theta)), t(\theta, s_i^*(\theta))\})$ maximizes the seller's expected profit, subject to (IC_{om}) and (IR), the proof of Theorem 1 implies $\boldsymbol{x}(\theta, s) = \boldsymbol{x}^*(\theta, s)$ for almost all $\theta \in [\underline{\theta}, \bar{\theta}]$ and $s \in S_i^*(\theta)$. Consequently, for any $i, j = 1, 2, \ldots, m$, we have

$$\begin{aligned}
& U(\eta, s_i^*(\eta); \theta, s_i^*(\eta)) - U(\eta, s_j^*(\eta); \theta, s_i^*(\eta)) \\
= & \mathbb{E}[v(q_i, \theta) - v(q_j, \theta) | s_i^*(\eta)] - \left(t(\eta, s_i^*(\eta)) - t(\eta, s_j^*(\eta)) \right)
\end{aligned} \tag{A7}$$

Under signal $s_i^*(\eta)$, $q_i \ge q_j$. If $v(.,.)$ has increasing differences in (q, θ) and $\theta > \eta$

or $v(.,.)$ has decreasing differences in (q, θ) and $\theta < \eta$, we have

$$\mathbb{E}[v(q_i, \theta) - v(q_j, \theta) | s_i^*(\eta)] \geq \mathbb{E}[v(q_i, \eta) - v(q_j, \eta) | s_i^*(\eta)] \tag{A8}$$

Substituting Inequality (A8) into Equation (A7), we have

$$\begin{aligned}
& U(\eta, s_i^*(\eta); \theta, s_i^*(\eta)) - U(\eta, s_j^*(\eta); \theta, s_i^*(\eta)) \\
\geq\ & \mathbb{E}[v(q_i, \eta) - v(q_j, \eta) | s_i^*(\eta)] - \big(t(\eta, s_i^*(\eta)) - t(\eta, s_j^*(\eta))\big) \\
=\ & U(\eta, s_i^*(\eta)) - U(\eta, s_j^*(\eta); \eta, s_i^*(\eta)) \\
\geq\ & 0,
\end{aligned}$$

as desired. □

Proof of Theorem 2: By Theorem 1, menu $(\{E^*(\theta)\}, \{\boldsymbol{x}^*(\theta, s_i^*(\theta)), t^*(\theta, s_i^*(\theta))\})$ maximizes the seller's expected profit, subject to (IC_{om}) and (IR), so it suffices to show this menu also respects (IC_{pr}). We prove the theorem in three steps: first, we show it is without loss to restrict to the case of $b(\theta) = \theta$; second, we show conditional on their truthful report of the type, the buyer has incentives to truthfully report the signal realization, that is, $U(\theta, s) \geq U(\theta, s'; \theta, s)$ for any $\theta \in [\underline{\theta}, \bar{\theta}]$ and $s, s' \in S^*(\theta)$; third, we show the set of (IC_{pr}) constraints is respected.

Step 1: For $\eta := b(\theta)$, let H be the distribution of η and h be the density function of η. Since the identity function is continuously differentiable, strictly increasing and weakly concave, it suffices to show η also has the monotone hazard rate property, that is, $\frac{1-H(\eta)}{h(\eta)}$ is non-increasing in η.

Since $b(.)$ is strictly increasing, b^{-1} exists and is strictly increasing. Moreover, $b(.)$ is continuously differentiable and weakly concave, so $b'(.) > 0$ almost everywhere and is non-increasing. Simple calculation shows

$$H(x) = F(b^{-1}(x)) \quad \text{and} \quad h(x) = f(b^{-1}(x))(b^{-1})'(x)$$

It follows that $\frac{1-H(\eta)}{h(\eta)} = \frac{1-F(b^{-1}(\eta))}{f(b^{-1}(\eta))(b^{-1})'(\eta)} = \frac{1-F(b^{-1}(\eta))}{f(b^{-1}(\eta))} b'(b^{-1}(\eta))$ is non-increasing.

Step 2: Let $v(q,\theta)=a(q)+\theta$. Since $U(\theta,s_j^*(\theta))\geq 0=U(\theta,s_0^*(\theta);\theta,s_j^*(\theta))$, it suffices to show the following two conditions.

- For any $i,j=1,2,\ldots,m$, $U(\theta,s_j^*(\theta))\geq U(\theta,s_i^*(\theta);\theta,s_j^*(\theta))$, that is,

$$\mathbb{E}[a(q_i)|s_i^*(\theta)]-\mathbb{E}[a(q_i)|s_j^*(\theta)]\geq\frac{\int_{\underline{\theta}}^{\theta}\pi^*(\eta,s_i^*(\eta))d\eta}{\pi^*(\theta,s_i^*(\theta))}-\frac{\int_{\underline{\theta}}^{\theta}\pi^*(\eta,s_j^*(\eta))d\eta}{\pi^*(\theta,s_j^*(\theta))}\quad\text{(A9)}$$

- For any $i=1,2,\ldots,m$, $0=U(\theta,s_0^*(\theta))\geq U(\theta,s_i^*(\theta);\theta,s_0^*(\theta))$, that is,

$$\mathbb{E}[a(q_i)|s_i^*(\theta)]-\mathbb{E}[a(q_i)|s_0^*(\theta)]\geq\frac{\int_{\underline{\theta}}^{\theta}\pi^*(\eta,s_i^*(\eta))d\eta}{\pi^*(\theta,s_i^*(\theta))}\quad\text{(A10)}$$

The buyer's virtual value $\varphi(q,\theta)=a(q)-c(q)+\theta-\frac{1-F(\theta)}{f(\theta)}$. By Assumption 2S, $(a-c)(.)$ is strictly increasing, so $(a-c)^{-1}$ exists. We define the modified threshold quality as follows:

$$\tilde{q}(\theta)=\begin{cases}(a-c)^{-1}\left(\frac{1-F(\underline{\theta})}{f(\underline{\theta})}-\theta\right) & \text{if } \theta<\underline{\theta}\\ (a-c)^{-1}\left(\frac{1-F(\theta)}{f(\theta)}-\theta\right) & \text{if } \theta\in[\underline{\theta},\bar{\theta}]\\ (a-c)^{-1}\left(\frac{1-F(\bar{\theta})}{f(\bar{\theta})}-\theta\right) & \text{if } \theta>\bar{\theta}\end{cases}$$

Then $\tilde{q}(.)$ is absolutely continuous and strictly decreasing, and its range covers $[\min_i \underline{q}_i,\max_i \bar{q}_i]$. Consequently, we can construct m mutually independent random variables η_i such that $\tilde{q}(\eta_i)=q_i$. Let $\theta_i^L=\tilde{q}^{-1}(\bar{q}_i)$, $\theta_i^H=\tilde{q}^{-1}(\underline{q}_i)$ and denote the distribution function of η_i by $\tilde{F}_i$. It follows that each $\tilde{F}_i$ is absolutely continuous on $[\theta_i^L,\theta_i^H]$. Now let $\theta_i^E=\max\{\underline{\theta},\theta_i^L\}$ be the effective lowest type for product i.

Consider the first term on the RHS of Inequality (A9).

$$\begin{aligned}&\frac{\int_{\underline{\theta}}^{\theta}\pi^*(\eta,s_i^*(\eta))d\eta}{\pi^*(\theta,s_i^*(\theta))}\\ =\quad&\frac{\int_{\underline{\theta}}^{\theta}\mathbb{P}(q_i\geq\max\{q_{-i},q^\star(\eta)\})d\eta}{\mathbb{P}(q_i\geq\max\{q_{-i},q^\star(\theta)\})}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\int_{\underline{\theta}}^{\theta} \mathbb{P}(\eta_i \le \min\{\eta_{-i}, \eta\}) d\eta}{\mathbb{P}(\eta_i \le \min\{\eta_{-i}, \theta\})} \\
&= \frac{\int_{\theta_i^E}^{\theta} \mathbb{P}(\eta_i \le \min\{\eta_{-i}, \eta\}) d\eta}{\mathbb{P}(\eta_i \le \min\{\eta_{-i}, \theta\})} \\
&= \frac{\int_{\theta_i^E}^{\theta} \left(\int_{\theta_i^L}^{\eta} (1 - \tilde{F}_{-i}(x)) d\tilde{F}_i(x) \right) d\eta}{\mathbb{P}(\eta_i \le \min\{\eta_{-i}, \theta\})} \\
&= \frac{\int_{\theta_i^L}^{\theta} (\theta - \eta)(1 - \tilde{F}_{-i}(\eta)) d\tilde{F}_i(\eta) - \int_{\theta_i^L}^{\theta_i^E} (\theta_i^E - \eta)(1 - \tilde{F}_{-i}(\eta)) d\tilde{F}_i(\eta)}{\mathbb{P}(\eta_i \le \min\{\eta_{-i}, \theta\})} \quad \text{(Fubini's theorem)} \\
&= \mathbb{E}[\theta - \eta_i | \eta_i \le \min\{\eta_{-i}, \theta\}] - \mathbb{E}[\mathbb{1}\{\eta_i \le \theta_i^E\}(\theta_i^E - \eta_i) | \eta_i \le \min\{\eta_{-i}, \theta\}] \\
&= \theta - \mathbb{E}[\max\{\eta_i, \underline{\theta}\} | \eta_i \le \min\{\eta_{-i}, \theta\}] \qquad \text{(A11)}
\end{aligned}
$$

Symmetrically, we can simplify the second term on the RHS of Inequality (A9) as

$$
\frac{\int_{\underline{\theta}}^{\theta} \pi^*(\eta, s_j^*(\eta)) d\eta}{\pi^*(\theta, s_j^*(\theta))} = \theta - \mathbb{E}[\max\{\eta_j, \underline{\theta}\} | \eta_j \le \min\{\eta_{-j}, \theta\}] \qquad \text{(A12)}
$$

Combining Equation (A11) with Equation (A12), we can simplify the RHS of Inequality (A9) as

$$
\begin{aligned}
& \frac{\int_{\underline{\theta}}^{\theta} \pi^*(\eta, s_i^*(\eta)) d\eta}{\pi^*(\theta, s_i^*(\theta))} - \frac{\int_{\underline{\theta}}^{\theta} \pi^*(\eta, s_j^*(\eta)) d\eta}{\pi^*(\theta, s_j^*(\theta))} \\
= \;& \mathbb{E}[\max\{\eta_j, \underline{\theta}\} | \eta_j \le \min\{\eta_{-j}, \theta\}] - \mathbb{E}[\max\{\eta_i, \underline{\theta}\} | \eta_i \le \min\{\eta_{-i}, \theta\}] \quad \text{(A13)}
\end{aligned}
$$

Next, we consider the LHS of Equation (A9). Since $c(.)$ is non-decreasing (Assumption 2S) and q_i are mutually independent (Assumption 5), we have for any i, j, θ,

$$
\mathbb{E}[c(q_i) | q_i \ge \max\{q_{-i}, q^\star(\theta)\}] \ge \mathbb{E}[c(q_i) | q_j \ge \max\{q_{-j}, q^\star(\theta)\}] \qquad \text{(A14)}
$$

Analogously, since $\frac{1-F(.)}{f(.)}$ and $\mathbb{1}\{. \le \theta_i^E\}(\theta_i^E - .)$ are both non-increasing and η_i

are mutually independent, we have for any i, j, θ,

$$\mathbb{E}\left[\frac{1-F(\eta_i)}{f(\eta_i)}|\eta_i \leq \min\{\eta_{-i}, \theta\}\right] \geq \mathbb{E}\left[\frac{1-F(\eta_i)}{f(\eta_i)}|\eta_j \leq \min\{\eta_{-j}, \theta\}\right] \tag{A15}$$

$$\mathbb{E}[\mathbb{1}\{\eta_i \leq \theta_i^E\}(\theta_i^E - \eta_i)|\eta_i \leq \min\{\eta_{-i}, \theta\}] \geq \mathbb{E}[\mathbb{1}\{\eta_i \leq \theta_i^E\}(\theta_i^E - \eta_i)|\eta_j \leq \min\{\eta_{-j}, \theta\}] \tag{A16}$$

It follows that

$$\begin{aligned}
& \mathbb{E}[a(q_i)|s_i^*(\theta)] - \mathbb{E}[a(q_i)|s_j^*(\theta)] \\
= \;& \mathbb{E}[a(q_i)|q_i \geq \max\{q_{-i}, q^\star(\theta)\}] - \mathbb{E}[a(q_i)|q_j \geq \max\{q_{-j}, q^\star(\theta)\}] \\
\geq \;& \mathbb{E}[a(q_i) - c(q_i)|q_i \geq \max\{q_{-i}, q^\star(\theta)\}] - \mathbb{E}[a(q_i) - c(q_i)|q_j \geq \max\{q_{-j}, q^\star(\theta)\}] \text{ (By A14)} \\
= \;& \mathbb{E}\left[\frac{1-F(\eta_i)}{f(\eta_i)} - \eta_i|\eta_i \leq \min\{\eta_{-i}, \theta\}\right] - \mathbb{E}\left[\frac{1-F(\eta_i)}{f(\eta_i)} - \eta_i|\eta_j \leq \min\{\eta_{-j}, \theta\}\right] \\
\geq \;& \mathbb{E}[\eta_i|\eta_j \leq \min\{\eta_{-j}, \theta\}] - \mathbb{E}[\eta_i|\eta_i \leq \min\{\eta_{-i}, \theta\}] \text{ (By A15)} \\
\geq \;& \mathbb{E}[\max\{\eta_i, \underline{\theta}\}|\eta_j \leq \min\{\eta_{-j}, \theta\}] - \mathbb{E}[\max\{\eta_i, \underline{\theta}\}|\eta_i \leq \min\{\eta_{-i}, \theta\}] \text{ (By A16)} \\
\geq \;& \mathbb{E}[\max\{\eta_j, \underline{\theta}\}|\eta_j \leq \min\{\eta_{-j}, \theta\}] - \mathbb{E}[\max\{\eta_i, \underline{\theta}\}|\eta_i \leq \min\{\eta_{-i}, \theta\}]
\end{aligned} \tag{A17}$$

Combining Inequality (A17) with Equation (A13), we get Inequality (A9).

Finally, we consider the LHS of Inequality (A10).

$$\begin{aligned}
& \mathbb{E}[a(q_i)|s_i^*(\theta)] - \mathbb{E}[a(q_i)|s_0^*(\theta)] \\
= \;& \mathbb{E}[a(q_i)|q_i \geq \max\{q_{-i}, q^\star(\theta)\}] - \mathbb{E}[a(q_i)|q_i \leq \min\{q_{-i}, q^\star(\theta)\}] \\
\geq \;& \mathbb{E}[a(q_i) - c(q_i)|q_i \geq \max\{q_{-i}, q^\star(\theta)\}] - \mathbb{E}[a(q_i) - c(q_i)|q_i \leq \min\{q_{-i}, q^\star(\theta)\}] \\
\geq \;& \mathbb{E}[a(q_i) - c(q_i) - \{a(q^\star(\theta)) - c(q^\star(\theta))\}|q_i \geq \max\{q_{-i}, q^\star(\theta)\}] \\
= \;& \mathbb{E}\left[\frac{1-F(\eta_i)}{f(\eta_i)} - \eta_i - \left\{\frac{1-F(\theta)}{f(\theta)} - \theta\right\}|\eta_i \leq \min\{\eta_{-i}, \theta\}\right] \\
\geq \;& \theta - \mathbb{E}[\eta_i|\eta_i \leq \min\{\eta_{-i}, \theta\}],
\end{aligned} \tag{A18}$$

where the first inequality is similar to (A14) and the third inequality is similar to (A15).

Combining Inequality (A18) with Equation (A11), we get Inequality (A10).

Step 3: Again let $v(q,\theta) = a(q) + \theta$. We need to show for any $\theta, \theta' \in [\underline{\theta}, \bar{\theta}]$,

$$\sum_{i=1}^{m} V_i(\theta) \geq \sum_{i=0}^{m} \max_{0 \leq j \leq m} U(\theta', s_j^*(\theta'); \theta, s_i^*(\theta'))\pi^*(\theta', s_i^*(\theta'))$$

By Lemma 3 and Step 2 above, we know for any $1 \leq i \leq m$,

$$\max_{0 \leq j \leq m} U(\theta', s_j^*(\theta'); \theta, s_i^*(\theta')) = \max\{U(\theta', s_i^*(\theta'); \theta, s_i^*(\theta')), 0\}$$

Consequently, it suffices to show

$$\sum_{i=1}^{m} V_i(\theta) \geq \max_{0 \leq j \leq m} U(\theta', s_j^*(\theta'); \theta, s_0^*(\theta'))\pi^*(\theta', s_0^*(\theta')) + \sum_{i=1}^{m} \max\{U(\theta', s_i^*(\theta'); \theta, s_i^*(\theta')), 0\}\pi^*(\theta', s_i^*(\theta'))$$

For $\theta \leq \theta'$ and any $1 \leq j \leq m$, by Assumption 1S and Step 2 above,

$$U(\theta', s_j^*(\theta'); \theta, s_0^*(\theta')) \leq U(\theta', s_j^*(\theta'); \theta', s_0^*(\theta')) \leq U(\theta', s_0^*(\theta')) = 0$$

Consequently, $\max_{0 \leq j \leq m} U(\theta', s_j^*(\theta'); \theta, s_0^*(\theta'))\pi^*(\theta', s_0^*(\theta')) = 0$ and the above inequality follows from the proof of Theorem 1.

It remains to consider $\theta > \theta'$. By Assumption 1S again, $U(\theta', s_i^*(\theta'); \theta, s_i^*(\theta')) \geq U(\theta', s_i^*(\theta')) \geq 0$ for any $1 \leq i \leq m$, so it suffices to show

$$\sum_{i=1}^{m} V_i(\theta) \geq U(\theta', s_j^*(\theta'); \theta, s_0^*(\theta'))\pi^*(\theta', s_0^*(\theta')) + \sum_{i=1}^{m} U(\theta', s_i^*(\theta'); \theta, s_i^*(\theta'))\pi^*(\theta', s_i^*(\theta'))$$

for any $1 \leq j \leq m$, which is equivalent to

$$\begin{aligned}
\sum_{i=1}^{m} \int_{\underline{\theta}}^{\theta} \pi^*(\eta, s_i^*(\eta))d\eta \quad &\geq \quad \left[\theta - \theta' + \mathbb{E}[a(q_j)|s_0^*(\theta')] - \mathbb{E}[a(q_j)|s_j^*(\theta')] + \frac{\int_{\underline{\theta}}^{\theta'} \pi^*(t, s_j^*(t))dt}{\pi^*(\theta', s_j^*(\theta'))}\right] \pi^*(\theta', s_0^*(\theta')) \\
&+ \quad \sum_{i=1}^{m} \left[\theta - \theta' + \frac{\int_{\underline{\theta}}^{\theta'} \pi^*(\eta, s_i^*(\eta))d\eta}{\pi^*(\theta', s_i^*(\theta'))}\right] \pi^*(\theta', s_i^*(\theta')) \qquad \text{(A19)}
\end{aligned}$$

Consider the derivative of the two sides of Equation (A19) with respect to θ, we have

$$\text{LHS}'(\theta) = \sum_{i=1}^{m} \pi^*(\theta, s_i^*(\theta)) \leq 1 = \text{RHS}'(\theta)$$

In other words, it suffices to show Equation (A19) holds for $\theta = \bar{\theta}$.

In establishing Inequality (A10) in Step 2, we proved the following inequality: for any j,

$$\mathbb{E}[a(q_j)|s_j^*(\theta')] - a(q^\star(\theta')) \geq \frac{\int_{\underline{\theta}}^{\theta'} \pi^*(\eta, s_j^*(\eta))d\eta}{\pi^*(\theta, s_j^*(\theta))}$$

Consequently, for any j,

$$\frac{\int_{\underline{\theta}}^{\theta'} \pi^*(\eta, s_j^*(\eta))d\eta}{\pi^*(\theta, s_j^*(\theta))} - \mathbb{E}[a(q_j)|s_j^*(\theta')] \leq -a(q^\star(\theta')) \tag{A20}$$

Combining Inequality (A20) with Inequality (A19) evaluated at $\theta = \bar{\theta}$, we know it suffices to show for any j,

$$\begin{aligned} \sum_{i=1}^{m} \int_{\underline{\theta}}^{\bar{\theta}} \pi^*(\eta, s_i^*(\eta))d\eta \quad &\geq \quad \left[\bar{\theta} - \theta' + \mathbb{E}[a(q_j)|s_0^*(\theta')] - a(q^\star(\theta'))\right] \pi^*(\theta', s_0^*(\theta')) \\ &+ \quad \sum_{i=1}^{m} \left[\bar{\theta} - \theta' + \frac{\int_{\underline{\theta}}^{\theta'} \pi^*(\eta, s_i^*(\eta))d\eta}{\pi^*(\theta', s_i^*(\theta'))}\right] \pi^*(\theta', s_i^*(\theta')) \end{aligned} \tag{A21}$$

Next, we prove the following inequality: for any j,

$$a(q^\star(\theta')) - \mathbb{E}[a(q_j)|s_0^*(\theta')] \geq \frac{\int_{\theta'}^{\bar{\theta}} \pi^*(\eta, s_0(\eta))d\eta}{\pi^*(\theta', s_0^*(\theta'))} \tag{A22}$$

Let $\eta_j \sim \tilde{F}[\theta_j^L, \theta_j^H]$ be as defined in Step 2, that is, $\tilde{q}(\eta_j) = q_j$. We have

$$\begin{aligned} &a(q^\star(\theta')) - \mathbb{E}[a(q_j)|s_0^*(\theta')] \\ \geq \quad &a(q^\star(\theta')) - c(q^\star(\theta')) - \mathbb{E}[a(q_j) - c(q_j)| \max_i q_i \leq q^\star(\theta')] \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\frac{1-F(\theta')}{f(\theta')} - \theta' - \frac{1-F(\eta_j)}{f(\eta_j)} + \eta_j | \theta' \le \min\{\eta_j, \eta_{-j}\}\right] \\
&\ge \mathbb{E}[\eta_j - \theta' | \theta' \le \min\{\eta_j, \eta_{-j}\}] \\
&= \frac{\left[1-\tilde{F}_{-j}(\theta')\right]\left[\int_{\theta'}^{\theta_j^H} (\eta_j - \theta') d\tilde{F}_j(\eta_j)\right]}{\pi^*(\theta', s_0^*(\theta'))} \\
&= \frac{\left[1-\tilde{F}_{-j}(\theta')\right]\left[\int_{\theta'}^{\theta_j^H} (1 - \tilde{F}_j(\eta_j)) d\eta_j\right]}{\pi^*(\theta', s_0^*(\theta'))} \\
&\ge \frac{\int_{\theta'}^{\theta_j^H} \left[1-\tilde{F}_j(\eta_j)\right]\left[1-\tilde{F}_{-j}(\eta_j)\right] d\eta_j}{\pi^*(\theta', s_0^*(\theta'))} \\
&= \frac{\int_{\theta'}^{\theta_j^H} \pi^*(\eta, s_0(\eta)) d\eta}{\pi^*(\theta', s_0^*(\theta'))},
\end{aligned} \tag{A23}
$$

where all of the three inequalities are by Assumption 5 (mutual independence) and the fact that $q_j \le q^\star(\theta')$ and $\eta_j \ge \theta'$ under signal $s_0^*(\theta')$.

For $\theta_j^H \ge \bar{\theta}$, we have $\frac{\int_{\theta'}^{\theta_j^H} \pi^*(\eta, s_0(\eta)) d\eta}{\pi^*(\theta', s_0^*(\theta'))} \ge \frac{\int_{\theta'}^{\bar{\theta}} \pi^*(\eta, s_0(\eta)) d\eta}{\pi^*(\theta', s_0^*(\theta'))}$.

For $\theta_j^H < \bar{\theta}$, $\pi^*(\theta, s_0(\theta)) = 0$ for any $\theta \in [\theta_H, \bar{\theta}]$, so $\frac{\int_{\theta'}^{\theta_j^H} \pi^*(\eta, s_0(\eta)) d\eta}{\pi^*(\theta', s_0^*(\theta'))} = \frac{\int_{\theta'}^{\bar{\theta}} \pi^*(\eta, s_0(\eta)) d\eta}{\pi^*(\theta', s_0^*(\theta'))}$.

Together with Inequality (A23), we get Inequality (A22).

Finally, applying Inequality (A22) to Inequality (A21), we know it suffices to show

$$
\int_{\underline{\theta}}^{\bar{\theta}} \left[\sum_{i=0}^m \pi^*(\eta, s_i^*(\eta))\right] d\eta \ge (\bar{\theta} - \theta') \sum_{i=0}^m \pi^*(\theta', s_i^*(\theta')) + \int_{\underline{\theta}}^{\theta'} \left[\sum_{i=0}^m \pi^*(\eta, s_i^*(\eta))\right] d\eta,
$$

which holds trivially because $\sum_{i=0}^m \pi^*(\eta, s_i^*(\eta)) = 1$ for any $\eta \in [\underline{\theta}, \bar{\theta}]$. $\square$

Proof of Proposition 2: Since the lowest type buyer would never purchase anything under this menu, $V(\underline{\theta}) = 0$. By the proof of Theorem 1, it suffices to show the optimal choice of the buyer given menu $(E^*, p_i = \mathbb{E}[v(q_i, \theta^\star) | s_i^*])$ coincides with the profit-maximizing allocation for every quality profile realization.

Since there is only one experiment, (IC_{om}) is irrelevant. We first show (IR) is respected at the profit-maximizing allocation under Assumptions 1 through 4.

Since $v(q,.)$ is strictly increasing in θ for any q, we have

$$\mathbb{E}[v(q_i,\theta)|s_i^*] - \mathbb{E}[v(q_i,\theta^\star)|s_i^*] \geq 0 \quad \text{for } \theta > \theta^\star$$

$$\mathbb{E}[v(q_i,\theta)|s_i^*] - \mathbb{E}[v(q_i,\theta^\star)|s_i^*] \leq 0 \quad \text{for } \theta \leq \theta^\star$$

Or the set of (IR) constraints is respected at the profit-maximizing allocation.

Next we show if Assumptions 5 also holds and $v(q,\theta)$ has non-decreasing differences in (q,θ), then (IC_{pr}) is respected at the profit-maximizing allocation.

For $\theta > \theta^\star$ and any i,j, we have

$$\begin{aligned}
& \mathbb{E}[v(q_i,\theta)|s_i^*] - \mathbb{E}[v(q_i,\theta^\star)|s_i^*] \\
\geq\ & \mathbb{E}[v(q_j,\theta)|s_i^*] - \mathbb{E}[v(q_j,\theta^\star)|s_i^*] \text{ (Non-decreasing differences)} \\
\geq\ & \max\left\{\mathbb{E}[v(q_j,\theta)|s_i^*] - \mathbb{E}[v(q_j,\theta^\star)|s_j^*], 0\right\} \text{ (Independence)}
\end{aligned}$$

For $\theta < \theta^\star$ and any i,j, we have

$$\begin{aligned}
& \mathbb{E}[v(q_j,\theta)|s_i^*] - \mathbb{E}[v(q_j,\theta^\star)|s_j^*] \\
\leq\ & \mathbb{E}[v(q_j,\theta)|s_j^*] - \mathbb{E}[v(q_j,\theta^\star)|s_j^*] \text{ (Independence)} \\
\leq\ & 0 \text{ (Monotonicity in } \theta)
\end{aligned}$$

In other words, the set of (IC_{pr}) constraints is respected at the profit-maximizing allocation. □

Proof of Proposition 3: We prove the proposition in two steps: first, we establish the "only if" part; second, we show when $v(q,\theta^\star(q)) \equiv v^\star$, menu $\left(E^{\text{full}}, p_i = v^\star\right)$ maximizes the seller's profit subject to (IC_{om}) and (IR). Moreover, this menu achieves the same profit when subject to (IC_{pr}) and (IR).

Step 1: We first show $q^\star(\theta)$ is strictly decreasing and continuous on $[\theta^\star(\max_i \bar{q}_i), \theta^\star(q_E)]$.

Take any $\theta^\star(\max_i \bar{q}_i) \leq \theta < \theta' \leq \theta^\star(q_E)$. Recall the virtual type $\varphi(q,\theta) = v(q,\theta) - c(q) - \frac{1-F(\theta)}{f(\theta)} v_\theta(q,\theta)$. By Assumptions 1 and 2, $\varphi(.,\theta)$ is continuous in q. Together with the definition of q_E, we must have $\varphi(q^\star(\theta),\theta) = 0$. Since $\varphi(q,.)$ is strictly increasing in θ, we have $\varphi(q^\star(\theta), \theta') > \varphi(q^\star(\theta), \theta) \geq 0$. By continuity in q again, we have $q^\star(\theta) > q^\star(\theta')$.

On the other hand, $\varphi(q, .)$ is continuous and strictly increasing in θ, so by the implicit function theorem (Kumagai, 1980), $\theta^\star(.)$ is continuous is q. Together with the assumption that $\theta^\star(.)$ is strictly decreasing, we have $q^\star(.)$ is continuous in θ.

Since p_i cannot depend on the signal realization, by an analogous argument to the proof of Lemma 2, it is without loss to consider $\tilde{E}(\theta) = E^*(\theta)$. For simplicity, we rank the products according to the $(\bar{q}_i, \underline{q}_i)$ lexicographic order, that is, $\bar{q}_1 \geq \bar{q}_2 \geq \cdots \geq \bar{q}_m$ and if $\bar{q}_i = \bar{q}_j$, $\underline{q}_i \geq \underline{q}_j$ if $i < j$.

Next we show $q^\star(\underline{\theta}) \geq \bar{q}_1$. Suppose to the contrary that $q^\star(\underline{\theta}) < \bar{q}_1$, which is depicted in Figure 3. Let $k = \max\{j : \bar{q}_j > q^\star(\underline{\theta})\}$. By the proof of Theorem 1, for menu $(\{E^*(\theta)\}, p_i)$ to maximize the seller's expected revenue subject to (IC_{om}) and (IR), we must have $V(\underline{\theta}) = 0$, where $V(\underline{\theta})$ represents the expected utility of buyer type $\underline{\theta}$. Consequently, we must have $\min_{1 \leq i \leq k} p_i > v(q^\star(\underline{\theta}), \underline{\theta})$. For instance, if $p_i \leq v(q^\star(\underline{\theta}), \underline{\theta})$, then $U(\underline{\theta}, s_i^*(\underline{\theta})) = \mathbb{E}[v(q_i, \theta) | s_i^*(\underline{\theta})] - p_i > 0$, which implies $V(\underline{\theta}) > 0$ (the buyer has to option to only purchase under signal realization $s_i^*(\underline{\theta})$).

Consider the incentive of buyer type $\theta > \underline{\theta}$. Recall that $V(\underline{\theta}; \theta)$ represents the expected utility of a type θ buyer, when they report and act as being type $\underline{\theta}$. We have

$$V(\theta) - V(\underline{\theta}; \theta) = \sum_{i=1}^{k} \int_{q_i \geq \max q_{-i}, q^\star(\theta) \leq q_i \leq q^\star(\underline{\theta})} [v(q_i, \theta) - p_i] \, dG(\boldsymbol{q})$$

Since $\min_{1 \leq i \leq k} p_i > v(q^\star(\underline{\theta}), \underline{\theta})$, $q^\star(.)$ is continuous in θ and $v(., .)$ is continuous in q and θ, we must have $V(\underline{\theta} + \varepsilon) - V(\underline{\theta}; \underline{\theta} + \varepsilon) < 0$ for $\varepsilon > 0$ small, which violates (IC_{om}).

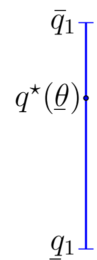


Figure 3: Proof of $q^\star(\underline{\theta}) \geq \bar{q}_1$

Next we show for any two products i, j such that $\bar{q}_i, \bar{q}_j > q_E$, we must have $p_i = p_j$. Consider the first two products. If $\bar{q}_2 \leq q_E$, then there is nothing to prove, because only product 1 will be allocated at the optimum. It remains to consider the two scenarios depicted in Figure 4.

- Scenario 1: $\bar{q}_1 = \bar{q}_2$.

  By the above argument, $\theta^\star(\bar{q}_1 - \varepsilon) > \underline{\theta}$ for any $\varepsilon > 0$. For the posted mechanism to maximize the seller's profit subject to (IC_{om}) and (IR), we must have $V(\theta^\star(\bar{q}_1)) = 0$, which implies $p_1 = p_2 = v(\bar{q}_1, \theta^\star(\bar{q}_1))$. If $\min\{p_1, p_2\} < v(\bar{q}_1, \theta^\star(\bar{q}_1))$, then $V(\theta^\star(\bar{q}_1)) > 0$. If $\max\{p_1, p_2\} > v(\bar{q}_1, \theta^\star(\bar{q}_1))$, then type $\theta^\star(\bar{q}_1)$ would not purchase under signal realization $s_1^*(\theta^\star(\bar{q}_1))$ or $s_2^*(\theta^\star(\bar{q}_1))$.

- Scenario 2: $\bar{q}_1 > \bar{q}_2$. By an analogous to Scenario 1, we must have $p_1 = v(\bar{q}_1, \theta^\star(\bar{q}_1))$. Next we argue $v(q, \theta^\star(q)) = p_1$ for any $q \in [\bar{q}_2, \bar{q}_1]$.

  Take any $\theta^\star(\bar{q}_2) \geq \theta > \theta' \geq \theta^\star(\bar{q}_1)$. ($\text{IC}_{\text{om}}$) implies

  $$V(\theta) \geq V(\theta'; \theta) \quad \text{and} \quad V(\theta; \theta') \geq V(\theta')$$

  Notice $q_1 > q_j$ for any j when $q_1 > \bar{q}_2$, so

  $$\int_{q^\star(\theta) \leq q_1 \leq q^\star(\theta')} (v(q_1, \theta) - p_1) dG_1(q_1) \geq 0 \geq \int_{q^\star(\theta) \leq q_1 \leq q^\star(\theta')} (v(q_1, \theta') - p_1) dG_1(q_1)$$

  In particular, $v(q^\star(\theta'), \theta) \geq p_1 \geq v(q^\star(\theta), \theta')$. Since $q^\star(.)$ is continuous, for $\theta \to \theta'$, we have $v(q_1^\star(\theta), \theta) = p_1$. Since $q^\star(.)$ is strictly decreasing in this range, we have $v(q, \theta^\star(q)) = p_1$ for any $q \in [\bar{q}_2, \bar{q}_1]$.

Now we argue that $p_2 = v(\bar{q}_2, \theta^\star(\bar{q}_2))$. If $p_2 > v(\bar{q}_2, \theta^\star(\bar{q}_2))$, then type $\theta^\star(\bar{q}_2)$ would not purchase product 2 under signal realization $s_2^*(\theta^\star(\bar{q}_2))$. If $p_2 < v(\bar{q}_2, \theta^\star(\bar{q}_2))$, then $V(\theta^\star(\bar{q}_2) - \varepsilon; \theta^\star(\bar{q}_2)) > V(\theta^\star(\bar{q}_2))$ for $\varepsilon > 0$ small. It follows that $p_1 = v(\bar{q}_2, \theta^\star(\bar{q}_2)) = p_2$.

(a) Scenario 1

(b) Scenario 2

Figure 4: Proof of $p_1 = p_2$

Let $k = \max\{i : \bar{q}_i > q_E\}$. Inductively, we can show $p_i = p_j = p$ for any $1 \leq i, j \leq k$.

Finally, we consider any $\theta^\star(q_E) \geq \theta > \theta' \geq \theta^\star(\bar{q}_1)$. ($\text{IC}_{\text{om}}$) implies

$$V(\theta) \geq V(\theta'; \theta) \quad \text{and} \quad V(\theta; \theta') \geq V(\theta')$$

Equivalently,

$$\begin{aligned} & \sum_{i=1}^{k} \int_{q_i \geq \max q_{-i}, q^\star(\theta) \leq q_i \leq q^\star(\theta')} v(q_i, \theta) dG(\boldsymbol{q}) \\ \geq \; & p \cdot \sum_{i=1}^{k} [\pi^*(\theta, s_i^*(\theta)) - \pi^*(\theta', s_i^*(\theta'))] \\ \geq \; & \sum_{i=1}^{k} \int_{q_i \geq \max q_{-i}, q^\star(\theta) \leq q_i \leq q^\star(\theta')} v(q_i, \theta') dG(\boldsymbol{q}) \end{aligned}$$

In particular, $v(q^\star(\theta'), \theta) \geq p \geq v(q^\star(\theta), \theta')$. Letting $\theta \to \theta'$, we have $v(q^\star(\theta), \theta) = p$

by continuity of $q^\star(.)$. By construction, the range of $q^\star(\theta)$ is $[q_E, \max_i \bar{q}_i]$. Moreover, $\theta^\star(.)$ is continuous and strictly decreasing, so $v(q, \theta^\star(q)) = p$ for any $q \in [q_E, \max_i \bar{q}_i]$.

Step 2: Similar to the proof of Proposition 2, it suffices to show $V(\underline{\theta}) = 0$ and the optimal choice of the buyer under menu $(E^{\text{full}}, p_i = v^\star)$ coincides with the profit-maximizing allocation for every quality profile realization. Specifically, we need to show for any $\boldsymbol{q} \in Q_i^* = \{\boldsymbol{q} \in Q : q_i \geq \max_{j<i} q_j, q_i > \max_{j>i} q_j\}$, the buyer would optimally purchase product i if and only if $\theta > \theta^\star(q_i)$.

First notice if $q_i < q^\star(\bar{\theta})$, then $q_E = q^\star(\bar{\theta})$ and $q_i < q_E$. We indeed have

$$v(q_i, \theta) - v^\star \leq v(q_i, \bar{\theta}) - v^\star = v(q_i, \bar{\theta}) - v(q_E, \bar{\theta}) < 0$$

or no buyer would purchase any product.

It remains to consider the case of $q_i \geq q^\star(\bar{\theta})$. For any $\theta > \theta^\star(q_i)$ and i, j, we have

$$v(q_i, \theta) - v^\star \geq v(q_j, \theta) - v^\star \quad \text{and}$$

$$v(q_i, \theta) - v^\star \geq v(q_i, \theta^\star(q_i)) - v^\star = 0,$$

where the first inequality is by the definition of Q_i^*.

For any $\theta < \theta^\star(q_i)$ and i, j, we have

$$v(q_j, \theta) - v^\star \leq v(q_i, \theta) - v^\star < v(q_i, \theta^\star(q_i)) - v^\star = 0$$

To conclude, under Assumptions 1 through 4, the set of (IR) and (IC_{pr}) constraints are respected at the profit-maximizing allocation. □

Proof of Proposition 4: We prove the proposition in two steps: first, we show if $\theta^\star(q)$ is not constant, then menu $(E^{\text{no}}, p = \mathbb{E}[v(q, \theta^\star)])$ does not maximize the seller's expected profit, subject to (IC_{om}) and (IR); second, we show if $\theta^\star(q) = \theta^\star$, then menu $(E^{\text{no}}, p = \mathbb{E}[v(q, \theta^\star)])$ maximizes the seller's expected profit, subject to (IC_{om}) and (IR). Moreover, this menu achieves the same profit, when subject to (IC_{pr}) and (IR)

Step 1: Since $\theta^\star(q)$ is not constant, we must have $\theta^\star(\bar{q}) < \theta^\star(\underline{q})$. By Step 1 of the proof of Proposition 3, for any $\theta^\star(\bar{q}) \leq \theta < \theta' \leq \theta^\star(\underline{q})$, we have $q^\star(\theta) > q^\star(\theta')$. Under menu $(E^{\text{no}}, p = \mathbb{E}[v(q, \theta^\star)])$, the buyer's surplus from purchase is given by $\mathbb{E}[v(q, \theta)] - \mathbb{E}[v(q, \theta^\star)]$, so the buyer's optimal choice induces the following allocation

$$x^{\text{no}}(\theta, q) = \begin{cases} 1 & \text{if } \theta > \theta^\star \\ 0 & \text{otherwise} \end{cases},$$

for any $q \in [\underline{q}, \bar{q}]$.

By Theorem 1, this allocation must be different from the profit-maximizing allocation for almost all $\theta \in (\theta^\star(\bar{q}), \theta^\star(\underline{q}))$ and a positive measure of $q \in [\underline{q}, \bar{q}]$. Let $\text{EP}(E^{\text{no}})$ be the expected profit from menu $(E^{\text{no}}, p = \mathbb{E}[v(q, \theta^\star)])$ and let $\text{EP}(E^*(\theta))$ be the expected profit characterized in Theorem 1. Similar to the proof of Proposition 1, we have $\text{EP}(E^{\text{no}}) < \text{EP}(E^*(\theta))$.

Step 2: When $\theta^\star(q) = \theta^\star$, we know

$$q^\star(\theta) = \begin{cases} \bar{q} & \text{if } \theta \leq \theta^\star \\ \underline{q} & \text{if } \theta > \theta^\star \end{cases}$$

Consequently, $x^{\text{no}}(\theta, q) = x^*(\theta, q)$ for almost all $\theta \in [\underline{\theta}, \bar{\theta}]$ and $q \in [\underline{q}, \bar{q}]$. Moreover, the expected utility of the lowest type $V(\underline{\theta}) = 0$. By the proof of Theorem 1, we have $\text{EP}(E^{\text{no}}) = \text{EP}(E^*(\theta))$.

Finally, $S^{\text{no}} = \{s_0\}$ is a singleton, so (IC_{pr}) is satisfied whenever (IR) is respected.

□

References

[1] Bergemann, D., Heumann, T. & Morris, S. (2023). Screening with persuasion. Working Paper.

[2] Bergemann, D. and Morrison, S. (2016). Information design, Bayesian persuasion, and Bayes correlated equilibrium. *American Economic Review: Papers and*

Proceedings, 106(5), 586–591.

[3] Bergemann, D. & Pesendorfer, M. (2007). Information structures in optimal auctions, *Journal of Economic Theory*, 137(1), 580–609.

[4] Blackwell, D. (1953). Equivalent comparisons of experiments, *The Annals of Mathematical Statistics*, 24(2), 265–272.

[5] Chen, W. (2019). Quality disclosure and price discrimination. Working Paper.

[6] Courty, L. & Hao, L. (2000). Sequential screening, *Review of Economic Studies*, 67, 697–717.

[7] Eső, P. & Szentes, B. (2007). Optimal information disclosure in auctions and the handicap auction, *Review of Economic Studies*, 74, 705–731.

[8] Guo, Y., Li, H. & Shi, X. (2022). Optimal discriminatory disclosure. Working Paper.

[9] Haghpanah, N. & Siegel R. (2022). The Limits of Multiproduct Price Discrimination, *American Economic Review: Insights*, 4(4), 443–458.

[10] Johnson, J.P. & Myatt, D. P. (2006). On the simple economics of advertising, market and product design, *American Economic Review*, 96(3), 756–784.

[11] Kamenica, E. & Gentzkow, M. (2011). Bayesian persuasion. *American Economic Review*, 101(6), 2590–2615.

[12] Kamenica, E. & Gentzkow, M. (2017). Bayesian persuasion with multiple senders and rich signal spaces. *Games and Economic Behavior*, 104, 411–429.

[13] Kolotilin, A. et al. (2017). Persuasion of a privately informed receiver. *Econometrica*, 85(6), 1949–1964.

[14] Krähmer, D. (2020). Information disclosure and full surplus extraction in mechanism design. *Journal of Economic Theory*, 187, 105020.

[15] Kumagai, S. (1980). An implicit function theorem: comment. *Journal of Optimization Theory and Applications*, 31(2), 285–288.

[16] Kumar, D. et al. (2022). Combining baloxavir marboxil with standard-of-care neuraminidase inhibitor in patients hospitalised with severe influenza (FLAGSTONE): a randomised, parallel-group, double-blind, placebo-controlled, superiority trial. *The Lancet infectious diseases*, 22(5), 718–730.

[17] Lewis, T.R. & Sappington, D.E.M. (1994). Supplying information to facilitate price discrimination. *International Economic Review*, 35(2), 309–327.

[18] Li, H. & Shi, X. (2017). Discriminatory information disclosure. *American Economic Review*, 107(11), 3363–3385.

[19] Maskin, E. & Riley, J. (1984). Monopoly with incomplete information. *RAND Journal of Economics*, 15(2), 171–196.

[20] Mussa, M. & Rosen, S. (1978). Monopoly and product quality. *Journal of Economic Theory*, 18(2), 301–317.

[21] Myerson, R.B. (1981). Optimal auction design. *Mathematics of Operations Research*, 6(1), 58–73.

[22] Ottaviani, M. & Prat, A. (2001). The value of public information in monopoly. *Econometrica*, 69(6), 1673–1683.

[23] Rayo, L. & Segal, I. (2010). Optimal information disclosure. *Journal of Political Economy*, 118(5), 949–987.

[24] Smolin, A. (2023). Disclosure and pricing of attributes. *RAND Journal of Economics*, 54(4), 570–597.

[25] Wei D. & Green, B. (2023). (Reverse) price discrimination with information design. *American Economic Journal: Microeconomics*, forthcoming.

[26] Yang, F. (2023). Nested bundling. Working Paper.